No-arbitrage with multiple-priors in discrete time

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Investors trading in a discrete-time, multi-period and multiple-prior financial market.

New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.
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- Revisit the so-called geometric and quantitative no-arbitrage conditions.
- Explicit two important examples where all these concepts are illustrated.
Framework and notations

Set of priors (similar to [Bouchard and Nutz, 2015])

- Sequence $(\Omega_t)_{1 \leq t \leq T}$ of Polish spaces. We denote

$$\omega^t = (\omega_1, \ldots, \omega_t) \in \Omega^t := \Omega_1 \times \cdots \times \Omega_t.$$
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\[ \omega^t = (\omega_1, \ldots, \omega_t) \in \Omega^t := \Omega_1 \times \cdots \times \Omega_t. \]

- **Assumption 1**: the one period set of priors $Q_{t+1} : \omega^t \in \Omega^t \mapsto \mathcal{P}(\Omega_{t+1})$ is a non-empty and convex valued random set s.t.

\[ \text{Graph}(Q_{t+1}) = \left\{ (\omega^t, P) \in \Omega^t \times \mathcal{P}(\Omega_{t+1}), \ P \in Q_{t+1}(\omega^t) \right\} \]

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  is an analytic set (continuous image of a Polish space).

- Jankov-von Neumann Theorem allows to construct the set of all possible priors $Q^T$.
  \[
  Q^T := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_T, Q_1 \in Q_1, q_{s+1} \in SK_{s+1}, q_{s+1}(:, \omega^s) \in Q_{s+1}(\omega^s) \forall \omega^s \forall 1 \leq s \leq T - 1 \},
  \]
  where $SK_{t+1}$ is the set of $B_c(\Omega^t) := \bigcap_{P \in \mathcal{P}(\Omega^t)} B_P(\Omega^t)$-meas. stochastic kernel on $\Omega_{t+1}$ given $\Omega^t$. 


The traded assets and strategies

- **Traded assets**: \( S := \{S_t, \ 0 \leq t \leq T\} \), \( d \)-dimensional process.
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- **Traded assets**: $S := \{S_t, \ 0 \leq t \leq T\}$, $d$-dimensional process.
- **Assumption 2**: $S$ is Borel-adapted.
- **Trading strategies**: $\phi := \{\phi_t, \ 1 \leq t \leq T\} \in \Phi$, universally-predictable $d$-dimensional process.
- **Trading is self-financing. Riskless asset’s price is constant 1.**

$$V_t^x,\phi = x + \sum_{s=1}^{t} \phi_s \Delta S_s.$$
No-arbitrage conditions

Monoprior NA(P)

\[ NA(P) : \ V_T^{0,\phi} \geq 0 \ \text{P-a.s. for some } \phi \in \Phi \ \Rightarrow \ V_T^{0,\phi} = 0 \ \text{P-a.s.} \]
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**Robust NA**

\[ NA(Q^{T}) : V_{T}^{0,\phi} \geq 0 \text{ } Q^{T}\text{-q.s. for some } \phi \in \Phi \Rightarrow V_{T}^{0,\phi} = 0 \text{ } Q^{T}\text{-q.s.} \]

See Bouchard and Nutz [2015]

\( N \subset \Omega^{T} \) is called a \( Q^{T}\)-polar set if \( \forall P \in Q^{T}, \exists A_{P} \in \mathcal{B}(X) \) such that \( P(A_{P}) = 0 \) and \( N \subset A_{P} \). Holds true \( Q^{T}\)-q.s. : outside a \( Q^{T}\)-polar set. \( Q^{T}\)-full measure set : complement of a \( Q^{T}\)-polar set.
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- Under this condition it is not even clear if there exists a model $P \in Q^T$ satisfying $NA(P)$. We will prove that this is in fact possible.
- Nevertheless, $Q^T$ might still contain some models that are not arbitrage free.
- An agent may not be able to delta-hedge a simple vanilla option using different levels of volatility in an arbitrage-free way.
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- $\text{sNA}(Q^{T})$ is useful to obtain tractable theorems for expected utility maximisation for unbounded function, see [Blanchard and Carassus, 2018] and [Rásonyi and Meireles-Rodrigues, 2018].
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- This definition seems also relevant in a continuous time setting for studying the no-arbitrage characterisation, see [Biagini et al., 2015].
No-arbitrage characterisations

**Weak NA**

\[ wNA(Q^T) : \exists P \in Q^T \text{ s.t. } NA(P). \]
No-arbitrage characterisations

Weak NA

\[ wNA(Q^T) : \exists P \in Q^T \text{ s.t. } NA(P). \]

The contraposition of the \(wNA(Q^T)\) condition is that for all models \(P \in Q^T\), there exists a strategy \(\phi_P\) such that \(V_T^{0,\phi_P} \geq 0\) \(P\)-a.s. and \(P(V_T^{0,\phi_P} > 0) > 0\).

A concrete example of a such model-dependent arbitrage is given in [Davis and Hobson, 2007].
Figure – Simple relations between the no-arbitrage definitions.
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2. There exists some $P^T \subset Q^T$ s.t.
   1. $P^T$ and $Q^T$ have the same polar-sets
   2. $sNA(P^T)$ holds true i.e. $NA(P)$ for all $P \in P^T$. 

Therefore, the set $P^T$ is defined recursively as follows:

$P_1^T := \{ \lambda P_*^T + (1 - \lambda) P, 0 < \lambda \leq 1, P \in Q_1^T \}$,

$P_{t+1}^T := \{ P_t^T \otimes (\lambda p_*^{t+1} + (1 - \lambda) q_{t+1}^T), 0 < \lambda \leq 1, P_t \in P_t^T, q_{t+1}^T(\omega) \in Q_{t+1}^T(\omega) \}$. 


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   - $\mathcal{P}^T$ and $Q^T$ have the same polar-sets
   - $sNA(\mathcal{P}^T)$ holds true i.e. $NA(P)$ for all $P \in \mathcal{P}^T$.

Let $P^*$ as in Theorem 11 below with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \cdots \otimes p_T^*$. The set $\mathcal{P}^T$ is defined recursively as follows:

$$\mathcal{P}^1 := \left\{ \lambda P_1^* + (1 - \lambda)P, \; 0 < \lambda \leq 1, \; P \in Q^1 \right\},$$
$$\mathcal{P}^{t+1} := \left\{ P_t \otimes (\lambda p_{t+1}^* + (1 - \lambda)q_{t+1}), \; 0 < \lambda \leq 1, \right\}$$

$$P_t \in \mathcal{P}^t, \; q_{t+1}(\cdot, \omega^t) \in Q_{t+1}(\omega^t) \; \forall \omega^t \in \Omega^t.$$
Applications

Equivalence between the $NA(Q^T)$ condition and the no-arbitrage condition introduced by [Bartl et al., 2019] which studies the problem of robust maximisation of expected utility using medial limits.
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Corollary

Assume that Assumptions 1. and 2. hold true. The following conditions are equivalent

- The $NA(Q^T)$ condition holds true.
- For all $Q \in Q^T$, there exists some $P \in \mathcal{P}^T$ such that $Q \ll P$ and such that $NA(P)$ holds true.
- For all $Q \in Q^T$, there exists some $P \in Q^T$ such that $Q \ll P$ and such that $NA(P)$ holds true.
Applications

Robust FTAP of [Bouchard and Nutz, 2015].

\[ \mathcal{R}^T := \{ P \in \mathcal{P}(\Omega^T), \exists Q' \in \mathcal{Q}^T, P \ll Q' \text{ and } P \text{ is a martingale measure} \}. \]

\[ \mathcal{K}^T := \{ P \in \mathcal{P}(\Omega^T), \exists Q' \in \mathcal{P}^T, P \sim Q' \text{ and } P \text{ is a martingale measure} \}. \]
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- The NA\((Q^{T})\) condition holds true.
- For all \(Q \in Q^{T}\), there exists some \(P \in \mathcal{K}^{T}\) such that \(Q \ll P\).
- For all \(Q \in Q^{T}\), there exists some \(P \in \mathcal{R}^{T}\) such that \(Q \ll P\).
Random Utility

\[ U : \Omega^T \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\} \text{ such that} \]

- for every \( x \in \mathbb{R} \), \( U(\cdot, x) : \Omega^T \rightarrow \mathbb{R} \cup \{\pm\infty\} \) is \( \mathcal{B}(\Omega^T) \)-measurable,
- for all \( \omega^T \in \Omega^T \), \( U(\omega^T, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\} \) is non-decreasing and concave on \((0, \infty)\)
- \( U(\cdot, x) = -\infty \), for all \( x < 0 \).
Applications

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- $U(\cdot, x) = -\infty$, for all $x < 0$.

Robust portfolio problem with initial wealth $x$

$u(x) := \sup_{\phi \in \Phi(x, U, Q^T)} \inf_{P \in Q^T} E_P U(\cdot, V^x_T, \phi(\cdot))$. \hspace{1cm} (1)

where $\Phi(x, U, Q^T)$ is the set of all strategies, s.t $V^x_T, \phi(\cdot) \geq 0$ $Q^T$-q.s. and $E_P U^+(\cdot, V^x_T, \phi(\cdot)) < \infty$ or $E_P U^-(\cdot, V^x_T, \phi(\cdot)) < \infty$ for all $P \in Q^T$. 
Applications

Assumption 3: We have that $U^+(\cdot, 1), U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ and $\Delta S_t, 1/\alpha_t^P \in \mathcal{W}_t$ for all $1 \leq t \leq T$ and $P \in \mathcal{P}^t$, where

$$\mathcal{W}_t := \bigcap_{r > 0} \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm \infty\}, \mathcal{B}(\Omega^t)\text{-measurable}, \sup_{P \in \mathcal{Q}^t} E_P |X|^r < \infty \right\}.$$
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Corollary

Assume that the $\text{NA}(Q^T)$ condition and Assumptions 1. and 2. hold true. Furthermore, assume that $U$ is either bounded from above or that Assumption 3. holds true. Then for all $x \geq 0$.

$$u(x) = u^P(x) := \sup_{\phi \in \Phi(x, U, \mathcal{P}^T)} \inf_{P \in \mathcal{P}^T} E_P U(\cdot, V_T^x, \phi(\cdot)).$$
Quantitative and geometric characterisation

Local NA

First part of [Bouchard and Nutz, 2015, Theorem 4.5]

**Theorem**

Assume that Assumptions 1 and 2 hold true. Then the following statements are equivalent:

1. The $\text{NA}(Q^T)$ condition holds true.
2. For all $0 \leq t \leq T - 1$, there exists a $Q^t$-full measure set $\Omega_{NA}^t \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega_{NA}^t$,

$$h\Delta S_{t+1}(\omega^t, \cdot) \geq 0 \quad Q_{t+1}(\omega^t)\text{-q.s.} \Rightarrow h\Delta S_{t+1}(\omega^t, \cdot) = 0 \quad Q_{t+1}(\omega^t)\text{-q.s.}$$
Measurability of the supports

Lemma

Let \( P \in Q^T \) with a fixed disintegration \( P := Q_1 \otimes q_2 \otimes \cdots \otimes q_T \). Under Assumptions 1 and 2, the following supports of the conditional distribution of \( \Delta S_{t+1}(\omega^t, \cdot) \)

\[
D_{t+1}^t(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{closed}, \ P_{t+1} \left( \Delta S_{t+1}(\omega^t, \cdot) \in A \right) = 1, \ \forall P_{t+1} \in Q_{t+1}(\omega^t) \right\}
\]

\[
D_{P}^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{closed}, \ q_{t+1} \left( \Delta S_{t+1}(\omega^t, \cdot) \in A, \omega^t \right) = 1 \right\},
\]

are non-empty, closed valued random set with graphs in \( \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d) \).
Geometric view in the spirit of [Jacod and Shiryaev, 1998, Theorem 3g]. Recall that \( \text{Ri}(C) = \{ y \in C, \exists \varepsilon > 0, \text{Aff}(C) \cap B(y, \varepsilon) \subset C \} \).

**Definition**

The geometric no-arbitrage condition holds true if for all \( 0 \leq t \leq T - 1 \), there exists some \( Q^t \)-full measure set \( \Omega_{gNA}^t \in \mathcal{B}_c(\Omega^t) \) such that for all \( \omega^t \in \Omega_{gNA}^t \), \( 0 \in \text{Ri} \left( \text{Conv} \left( D^{t+1} \right) \right) (\omega^t) \). In this case for all \( \omega^t \in \Omega_{gNA}^t \), there exists \( \varepsilon_t(\omega^t) > 0 \) such that

\[
B(0, \varepsilon_t(\omega^t)) \cap \text{Aff} \left( D^{t+1} \right) (\omega^t) \subset \text{Conv} \left( D^{t+1} \right) (\omega^t).
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Geometric view in the spirit of [Jacod and Shiryaev, 1998, Theorem 3g]].
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\]

- The geometric (local) no-arbitrage condition is indeed practical: it allows to check whether the (global) \( \text{NA}(Q^T) \) condition holds true or not.
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\]

- The geometric (local) no-arbitrage condition is indeed practical: it allows to check whether the (global) NA\( (Q^T) \) condition holds true or not.
- As \( Q_{t+1} \) and \( \Delta S_{t+1} \) are given one gets \( \text{Ri} \left( \text{Conv}(D^{t+1}) \right)(\cdot) \) and it is easy to check whether 0 is in it or not.
Quantitative NA

Quantitative view the spirit of [Rásonyi and Stettner, 2005, Proposition 3.3]

Definition

The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T - 1$, there exists some $Q^t$-full measure set $\Omega_{qNA}^t \in B_c(\Omega^t)$ such that for all $\omega^t \in \Omega_{qNA}^t$, there exists $\beta_t(\omega^t), \kappa_t(\omega^t) \in (0, 1]$ such that for all $h \in \text{Aff} \left( D^{t+1} \right) (\omega^t), h \neq 0$ there exists $P_h \in Q_{t+1}(\omega^t)$ satisfying

$$P_h \left( \frac{h}{|h|} \Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t) \right) \geq \kappa_t(\omega^t).$$
Quantitative NA

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$$P_{h} \left( \frac{h}{|h|} \Delta S_{t+1}(\omega^{t}, \cdot) < -\beta_{t}(\omega^{t}) \right) \geq \kappa_{t}(\omega^{t}).$$

- One risky asset and one period: there exists a prior for which the price of the risky asset increases enough and another one for which it decreases, $P^{\pm}(\mp \Delta S(\cdot) < -\alpha) > \alpha$ where $\alpha > 0$. 
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- One risky asset and one period: there exists a prior for which the price of the risky asset increases enough and an other one for which it decreases, $P^\pm (\mp \Delta S(\cdot) < -\alpha) > \alpha$ where $\alpha > 0$.
- The probability measure depends of the strategy.
Quantitative and geometric NA

Quantitative NA

- $\beta_t(\omega^t)$ provides information on $D^{t+1}(\omega^t)$ while $\kappa_t(\omega^t)$ provides information on $Q_{t+1}(\omega^t)$. 
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Quantitative NA

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- Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.

- When $\text{Dom}(U) = (0, \infty)$ it provides natural bounds for the one step strategies or for $U(V_T^x, \Phi)$, see [Blanchard and Carassus, 2018].
Quantitative NA

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- Used to prove the existence of the optimal strategy but could also be used to compute it numerically.
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• Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.

• When $\text{Dom}(U) = (0, \infty)$ it provides natural bounds for the one step strategies or for $U(V^{x,\Phi}_T)$, see [Blanchard and Carassus, 2018].

• Used to prove the existence of the optimal strategy but could also be used to compute it numerically.

• Explicit values for $\beta_t$ and $\kappa_t$ are given.
**Theorem**

*Assume that Assumptions 1 and 2 hold true. Then the $NA(Q^T)$ condition, the geometric no-arbitrage and the quantitative no-arbitrage are equivalent and one can choose for all $0 \leq t \leq T - 1$*

$$
\Omega_{t NA} = \Omega_{t qNA} = \Omega_{t gNA}
$$

*and $\beta_t = \varepsilon_t / 2$.***
Theorem

Assume that Assumptions 1 and 2 hold true. Then the \( NA(Q^T) \) condition, the geometric no-arbitrage and the quantitative no-arbitrage are equivalent and one can choose for all \( 0 \leq t \leq T - 1 \)

\[
\Omega^t_{NA} = \Omega^t_{qNA} = \Omega^t_{gNA}
\]

and \( \beta_t = \varepsilon_t / 2 \).

Proposition

Assume that Assumptions 1 and 2 hold true. Under one of the no-arbitrage conditions one can choose an universally measurable version of \( \varepsilon_t \) and \( \beta_t \).
Probability measure $P^*$

- $wNA(Q^T)$ does not imply $NA(Q^T)$ condition.
Probability measure $P^*$

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- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \rightarrow \mathbb{R}$. 

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- One period model with two risky assets $S^i_0 = 0$ and $S^i_1 : \Omega \to \mathbb{R}$.
- Let $P_1$ s.t. $P_1(\Delta S^1_1 \geq 0) = 1$, $P_1(\Delta S^1_1 > 0) > 0$. 


Second Main result

Probability measure $P^*$

- $wNA(Q^T)$ does not imply $NA(Q^T)$ condition.
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- Let $P_1$ s.t. $P_1(\Delta S_1^1 \geq 0) = 1$, $P_1(\Delta S_1^1 > 0) > 0$.
- Let $P_2$ s.t. $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm \Delta S_1^2 > 0) > 0$. 

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- Let $P_2$ s.t. $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm \Delta S_1^2 > 0) > 0$.
- $Q = \{\lambda P_1 + (1 - \lambda)P_2, \ 0 < \lambda \leq 1\}$. 
Probability measure $P^*$

- $wNA(Q^T)$ does not imply $NA(Q^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
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- $Q = \{\lambda P_1 + (1 - \lambda)P_2, \ 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(Q)$ hold true.
Probability measure $P^*$

- $wNA(Q^T)$ does not imply $NA(Q^T)$ condition.
- One period model with two risky assets $S^i_0 = 0$ and $S^i_1 : \Omega \rightarrow \mathbb{R}$.
- Let $P_1$ s.t. $P_1(\Delta S^1_1 \geq 0) = 1$, $P_1(\Delta S^1_1 > 0) > 0$.
- Let $P_2$ s.t. $P_2(\Delta S^1_1 = 0) = 1$, $P_2(\pm \Delta S^2_1 > 0) > 0$.
- $Q = \{\lambda P_1 + (1 - \lambda)P_2, \ 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(Q)$ hold true.
- $NA(Q)$ condition does not hold true: Let $h = (1, 0)$. Then $h\Delta S^1_1 \geq 0$ $Q$-q.s. but $P_1(h\Delta S^1_1 > 0) > 0$. 

Note that $Aff(D_{P_2}) = \mathbb{R}^2$ and $Aff(D_{P_2}) = \{(0,0)\} \times \mathbb{R}$. 
Probability measure $P^*$

- $wNA(Q^T)$ does not imply $NA(Q^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \rightarrow \mathbb{R}$.
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- $Q = \{\lambda P_1 + (1 - \lambda)P_2, 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(Q)$ hold true.
- $NA(Q)$ condition does not hold true: Let $h = (1, 0)$. Then $h\Delta S_1^1 \geq 0$ $Q$-q.s. but $P_1(h\Delta S_1 > 0) > 0$.
- Note that $Aff(D) = \mathbb{R}^2$ and $Aff(D_{P_2}) = \{0\} \times \mathbb{R}$.
Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- \( NA(Q^T) \) holds true.
Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(Q^T)$ holds true.
- There exists some $P^* \in Q^T$ such that for all $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t_{NA}$
**Theorem**

*Assume that Assumptions 1. and 2. hold true. TFAE*

- $NA(Q^T)$ holds true.
- There exists some $P^* \in Q^T$ such that for all $0 \leq t \leq T - 1$,
  * $\omega^t \in \Omega_{NA}^t$
  * $Aff(D_{P*}^{t+1}) (\omega^t) = Aff(D^{t+1}) (\omega^t)$
Theorem

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  - $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1})) (\omega^t)$. 
Theorem

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- $NA(P^*)$ condition holds true and even more.
Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(Q^T)$ holds true.
- There exists some $P^* \in Q^T$ such that for all $0 \leq t \leq T - 1,
  \omega^t \in \Omega^t_{NA}$
  - $Aff(D^{t+1}_{P^*}) (\omega^t) = Aff(D^{t+1}) (\omega^t)$
  - $0 \in Ri(Conv(D^{t+1}_{P^*})) (\omega^t)$.

- $NA(P^*)$ condition holds true and even more.
- The condition $Aff(D^{t+1}_{P^*})(\cdot) = Aff(D^{t+1})(\cdot) Q^t$-q.s. is necessary
  (see the preceding counterexample).
Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- \( \text{NA}(Q^T) \) holds true.
- There exists some \( P^* \in Q^T \) such that for all \( 0 \leq t \leq T - 1 \), \( \omega^t \in \Omega^t_{\text{NA}} \)
  - \( \text{Aff}(D^{t+1}_{P^*}) (\omega^t) = \text{Aff}(D^{t+1}) (\omega^t) \)
  - \( 0 \in \text{Ri} \left( \text{Conv}(D^{t+1}_{P^*}) \right) (\omega^t) \).

- \( \text{NA}(P^*) \) condition holds true and even more.
- The condition \( \text{Aff}(D^{t+1}_{P^*}) (\cdot) = \text{Aff}(D^{t+1}) (\cdot) \) \( Q^t \text{-q.s.} \) is necessary (see the preceding counterexample).
- Other counterexample if \( 0 \in \text{Ri} \left( \text{Conv}(D^{t+1}_{P^*}) \right) (\cdot) \) \( P^*_t \text{-p.s.} \) instead of \( 0 \in \text{Ri} \left( \text{Conv}(D^{t+1}_{P^*}) \right) (\cdot) \) \( Q^t \text{-q.s.} \).
Second Main result

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- \( NA(Q^T) \) holds true.
- There exists some \( P^* \in Q^T \) such that for all \( 0 \leq t \leq T - 1 \), \( \omega^t \in \Omega_{NA}^t \)
  - \( Aff(D^{t+1}_{P^*}) (\omega^t) = Aff(D^{t+1}) (\omega^t) \)
  - \( 0 \in Ri(Conv(D^{t+1}_{P^*})) (\omega^t) \).

- \( NA(P^*) \) condition holds true and even more.
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- \( P^* \) was used to build \( \mathcal{P}^T \).
Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(Q^T)$ holds true.
- There exists some $P^* \in Q^T$ such that for all $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t_{NA}$
  - $\text{Aff}(D_{P^*}^{t+1})(\omega^t) = \text{Aff}(D^{t+1})(\omega^t)$
  - $0 \in \text{Ri}(\text{Conv}(D_{P^*}^{t+1}))(\omega^t)$.

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- $P^*$ was used to build $\mathcal{P}^T$.
- The probability measure $P^*$ of Theorem 11 is not unique.
Complement [Oblój and Wiesel, 2018, Theorem 3.1] which makes the link with the quasi-sure setting.
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• The existence of $P^*$ show that $NA(Q^T)$ implies that (an adaptation of) [Rásonyi and Meireles-Rodrigues, 2018, Assumption 2.1] and thus [Rásonyi and Meireles-Rodrigues, 2018, Theorem 3.7] which shows the existence in the problem of maximisation of expected utility for bounded function defined on the whole real line works under $NA(Q^T)$. 
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One can choose $P^*$ as common probability measure in the quantitative definition of NA.

Allows to find universally measurable version of $\kappa_t$. 
Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure $\hat{P} \in Q^T$. Then

- The $NA(\hat{P})$ and the $NA(Q^T)$ conditions are equivalent.
Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure $\hat{P} \in Q^T$. Then

- **The** $NA(\hat{P})$ **and the** $NA(Q^T)$ **conditions are equivalent.**
- **One can choose** $P^* = \hat{P}$ in $\mathcal{P}^T$. 
Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure \( \hat{P} \in Q^T \). Then

- The \( \text{NA}(\hat{P}) \) and the \( \text{NA}(Q^T) \) conditions are equivalent.
- One can choose \( P^* = \hat{P} \) in \( P^T \).

Proposition

Assume that Assumption 2. holds true and that there exists

1. some \( \tilde{P} \in Q^T \)
2. some \( 0 \leq \tilde{t} \leq T - 1 \) and some \( \Omega_{\tilde{t}}^N \in \mathcal{B}_c(\Omega_{\tilde{t}}) \) such that
   - \( \tilde{P}^{\tilde{t}}(\Omega_{\tilde{t}}^N) > 0 \)
   - \( Q_{\tilde{t}+1}(\omega^{\tilde{t}}) \) is not dominated for all \( \omega^{\tilde{t}} \in \Omega_{\tilde{t}}^N \).

Then \( Q^T \) is not dominated.
Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$. 
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$S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where $Y_{t+1}$ Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$. 

Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.

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- Assumption 1. is verified. Let

$$B_{t+1}(\omega^t) := \{p\delta_u + (1 - p)\delta_d, \ p_t(\omega^t) \leq p \leq P_t(\omega^t), \ u_t(\omega^t) \leq u \leq U_t(\omega^t), \ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},$$
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$$

where $P_t, U_t, D_t$ are Borel-measurable r.v. s.t.
• Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.

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where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.

• $p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1, \ P_t(\omega^t) > 0$.
Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.

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where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.

- $p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1, \ P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.
Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.

$S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where $Y_{t+1}$ Borel-measurable r.v. s.t.
$Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.

Assumption 1. is verified. Let

$$B_{t+1}(\omega^t) := \{p\delta_u + (1 - p)\delta_d, \, p_t(\omega^t) \leq p \leq P_t(\omega^t),$$
$$u_t(\omega^t) \leq u \leq U_t(\omega^t), \, d_t(\omega^t) \leq d \leq D_t(\omega^t)\},$$

where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.

- $p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1, \, P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.

Then

$$Q_{t+1}(\omega^t) := \text{Conv} \left( \{Q \in \mathcal{P}(\Omega_{t+1}), \, Q(Y_{t+1} \in \cdot) \in B_{t+1}(\omega^t)\} \right),$$
- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where $Y_{t+1}$ Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let
  \[ B_{t+1}(\omega^t) := \{ p \delta_u + (1 - p) \delta_d, \quad p_t(\omega^t) \leq p \leq P_t(\omega^t), \]
  \[ u_t(\omega^t) \leq u \leq U_t(\omega^t), \quad d_t(\omega^t) \leq d \leq D_t(\omega^t) \}, \]
where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1$, $P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.
- Then
  \[ Q_{t+1}(\omega^t) := \text{Conv} \left( \{ Q \in \mathcal{P}(\Omega_{t+1}), \quad Q(Y_{t+1} \in \cdot) \in B_{t+1}(\omega^t) \} \right), \]
Usual binomial model corresponds: $p_t = P_t = p$, $u_t = U_t = u$ and $d_t = D_T = d$ where $0 < p < 1$, $d < 1 < u$.  

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where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.

$p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1, P_t(\omega^t) > 0$

$0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.

Then

$$Q_{t+1}(\omega^t) := \text{Conv} \left( \{Q \in \mathcal{P}(\Omega_{t+1}), \ Q(Y_{t+1} \in \cdot) \in B_{t+1}(\omega^t)\} \right),$$

Usual binomial model corresponds : $p_t = P_t = p$, $u_t = U_t = u$ and $d_t = D_T = d$ where $0 < p < 1$, $d < 1 < u$.

Assumption 2. holds true.
$S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
\[ S_{t+1} - S_t = S_t(Y_{t+1} - 1) \text{ and } 0 < d_t(\omega^t) < 1 < U_t(\omega^t) \]

\[ \text{Conv}(D^{t+1})(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)] \]
• $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$

• $\text{Conv}(D^{t+1})(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.

• $NA(Q^T)$ holds true: $0 \in \text{Ri} \left(\text{Conv}(D^{t+1})\right)(\omega^t)$ for all $\omega^t \in \Omega^t$. 

\begin{itemize}
  \item $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
  \item $\text{Conv} \left( D^{t+1} \right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.
  \item $NA(Q^T)$ holds true: $0 \in \text{Ri} \left( \text{Conv} \left( D^{t+1} \right) \right)(\omega^t)$ for all $\omega^t \in \Omega^t$.
  \item If for instance $u_t(\omega^t) < 1$ for all $\omega^t$, $\exists a_t(\omega^t) \in [u_t(\omega^t), 1)$. Let for $\pi_t(\omega^t) \in [p_t(\omega^t), P_t(\omega^t)]$
    \begin{align*}
      q_{t+1}(\Delta S_{t+1} \in \cdot, \omega^t) &= \pi_t(\omega^t)\delta_{a_t(\omega^t)}(\cdot) + (1 - \pi_t(\omega^t))\delta_{d_t(\omega^t)}(\cdot) \\
      Q &= Q_1 \otimes \cdots \otimes q_t \in Q^T \\
      \text{Conv} \left( D_{Q}^{t+1} \right)(\omega^t) &= [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(a_t(\omega^t) - 1)]
    \end{align*}
\end{itemize}
• $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
• $\text{Conv} \left( D^{t+1} \right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)].$
• $\text{NA}(Q^T)$ holds true: $0 \in \text{Ri} \left( \text{Conv} \left( D^{t+1} \right) \right)(\omega^t)$ for all $\omega^t \in \Omega^t.$
• If for instance $u_t(\omega^t) < 1$ for all $\omega^t$, $\exists a_t(\omega^t) \in [u_t(\omega^t), 1)$. Let for $\pi_t(\omega^t) \in [p_t(\omega^t), P_t(\omega^t)]$

$$q_{t+1}(\Delta S_{t+1} \in \cdot, \omega^t) = \pi_t(\omega^t)\delta_{a_t(\omega^t)}(\cdot) + \left(1 - \pi_t(\omega^t)\right)\delta_{d_t(\omega^t)}(\cdot)$$

$$Q = Q_1 \otimes \cdots \otimes q_t \in Q^T$$

$$\text{Conv} \left( D_Q^{t+1} \right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(a_t(\omega^t) - 1)]$$

• $0 \notin \text{Ri} \left( \text{Conv} \left( D_Q^{t+1} \right) \right)(\omega^t)$ for all $\omega^t \in \Omega^t$ and both the $\text{NA}(Q)$ and $\text{sNA}(Q^T)$ conditions fail.
- Explicit expressions for $\varepsilon_t$, $\beta_t$ and $\kappa_t$

\[
\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t) - 1}{2}, \frac{1 - d_t(\omega^t)}{2}\right) > 0,
\]
\[
\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t) + P_t(\omega^t)}{2}, 1 - \frac{p_t(\omega^t) + P_t(\omega^t)}{2}\right) > 0.
\]
Explicit expressions for $\varepsilon_t, \beta_t$ and $\kappa_t$

$$
\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min \left( \frac{U_t(\omega^t) - 1}{2}, \frac{1 - d_t(\omega^t)}{2} \right) > 0,
$$

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$N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.
**Explicit expressions for \( \varepsilon_t, \beta_t \) and \( \kappa_t \)**

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\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min \left( \frac{U_t(\omega^t) - 1}{2}, \frac{1 - d_t(\omega^t)}{2} \right) > 0,
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- \( N > 1 \) and \( M \geq 1 \) are fixed and allows to get sharper bounds.
- The (Borel) measurability of \( \varepsilon_t, \beta_t \) and \( \kappa_t \) are clear.
• Explicit expressions for $\varepsilon_t$, $\beta_t$ and $\kappa_t$

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t) - 1}{2}, \frac{1 - d_t(\omega^t)}{2}\right) > 0,$$

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• $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.

• The (Borel) measurability of $\varepsilon_t$, $\beta_t$ and $\kappa_t$ are clear.

• Let $\bar{\pi}_t(\omega^t) := \frac{p_t(\omega^t) + P_t(\omega^t)}{2} \in (0, 1)$ and $a^\pm, b^\pm$ be chosen such that

$$a^+_t(\omega^t) := U_t(\omega^t) > 1, \quad b^+_t(\omega^t) := \min\left(D_t(\omega^t) + \frac{d_t(\omega^t) + 1}{2}\right) < 1,$$

$$a^-_t(\omega^t) := \max\left(u_t(\omega^t) + \frac{U_t(\omega^t) + 1}{2}\right) > 1, \quad b^-_t(\omega^t) := d_t(\omega^t) < 1,$$

$$r^+_t(\omega^t) := 1 - \bar{\pi}_t(\omega^t), \quad b^-_t(\omega^t) = d_t(\omega^t) < 1,$$

$$r^*_t(\omega^t) := \frac{1}{2} \left(r^+_t(\omega^t) + r^-_t(\omega^t)\right) \in \mathcal{B}_{t+1}(\omega^t),$$

$$p^*_t(Y_{t+1} \in \omega^t) := r^*_t(\omega^t) \in \mathcal{Q}_{t+1}(\omega^t).$$
Explicit expressions for $\varepsilon_t$, $\beta_t$ and $\kappa_t$

\[
\begin{align*}
\varepsilon_t(\omega^t) &= \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t) - 1}{2}, \frac{1 - d_t(\omega^t)}{2}\right) > 0, \\
\kappa_t(\omega^t) &= \frac{1}{M} \min\left(\frac{p_t(\omega^t) + P_t(\omega^t)}{2}, 1 - \frac{p_t(\omega^t) + P_t(\omega^t)}{2}\right) > 0.
\end{align*}
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- $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.
- The (Borel) measurability of $\varepsilon_t$, $\beta_t$ and $\kappa_t$ are clear.
- Let $\bar{\pi}_t(\omega^t) := \frac{p_t(\omega^t) + P_t(\omega^t)}{2} \in (0, 1)$ and $a^{\pm}, b^{\pm}$ be chosen such that

\[
\begin{align*}
a_t^+(\omega^t) &:= U_t(\omega^t) > 1, \quad b_t^+(\omega^t) := \min\left(D_t(\omega^t), \frac{d_t(\omega^t) + 1}{2}\right) < 1, \\
a_t^-(\omega^t) &:= \max\left(u_t(\omega^t), \frac{U_t(\omega^t) + 1}{2}\right) > 1, \quad b_t^-(\omega^t) := d_t(\omega^t) < 1,
\end{align*}
\]

\[
\begin{align*}
r_t^{\pm}(\cdot, \omega^t) &:= \bar{\pi}_t(\omega^t)\delta_{a_t^{\pm} \omega^t}(\cdot) + (1 - \bar{\pi}_t(\omega^t))\delta_{b_t^{\pm}(\omega^t)}(\cdot) \in \mathcal{B}_{t+1}(\omega^t), \\
 r_t^*(\cdot, \omega^t) &:= \frac{1}{2} (r_t^+(\cdot, \omega^t) + r_t^-(\cdot, \omega^t)) \in \mathcal{B}_{t+1}(\omega^t),
\end{align*}
\]

\[
p_t^*(Y_{t+1} \in \cdot, \omega^t) := r_t^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)
\]

\[
p_t^*(\pm \Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t), \omega^t) \geq \kappa_t(\omega^t).
\]
Choose
Choose

\[ P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in Q^T. \]

0 \in \text{Ri} \left( \text{Conv} \left( D_{P^*}^{t+1} \right) \right) (\omega^t)

and that \( \text{Aff} \left( D_{P^*}^{t+1} \right) (\omega^t) = \text{Aff} \left( D^{t+1} \right) (\omega^t) \) for all \( \omega^t \).
Choose

\[ P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in Q^T. \quad 0 \in \text{Ri} \left( \text{Conv}(D_{P^*}^{t+1}) \right) (\omega^t) \]

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Note that \( P^* \) is not unique.
Choose

\[ P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in \mathcal{Q}^T. \quad 0 \in \text{Ri}\left(\text{Conv}(D_{P^*}^{t+1})\right)(\omega^t) \]
and that \( \text{Aff}\left(\text{Conv}(D_{P^*}^{t+1})\right)(\omega^t) = \text{Aff}\left(\text{Conv}(D_t^{t+1})\right)(\omega^t) \) for all \( \omega^t \).

Note that \( P^* \) is not unique.

Finally, if for some \( 0 \leq t \leq T - 1, \omega^t \in \Omega^t, u_t(\omega^t) < U_t(\omega^t) \) or \( d_t(\omega^t) < D_t(\omega^t) \) the set \( \mathcal{Q}_{t+1}(\omega^t) \) is non-dominated and \( \mathcal{Q}^T \) is also non-dominated.
Choose

$P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in Q^T. 0 \in \text{Ri} \left( \text{Conv} \left( D_{P^*}^{t+1} \right) \right)(\omega^t)$

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Finally, if for some $0 \leq t \leq T - 1$, $\omega^t \in \Omega^t$, $u_t(\omega^t) < U_t(\omega^t)$ or $d_t(\omega^t) < D_t(\omega^t)$ the set $Q_{t+1}(\omega^t)$ is non-dominated and $Q^T$ is also non-dominated.

Indeed, if not, any dominating measure would have an uncountable number of atoms.
Conclusion

- We have understood in details the quasi-sure no arbitrage condition and studied the link with different types of robust no-arbitrage conditions (local or global) in discrete time.
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Conclusion

- We have understood in details the quasi-sure no arbitrage condition and studied the link with different types of robust no-arbitrage conditions (local or global) in discrete time.
- Our main result gives the existence of a set of priors having the same polar sets than the original one and such each priors is arbitrage free.
- We give concrete examples where all the quantities appearing in the different definitions and characterizations of NA are explicit.


