No-arbitrage with multiple-priors in discrete time

Model Uncertainty in Risk Management, 31/01/2020

Laurence Carassus, Research Center, L. de Vinci Pôle universitaire and URCA.
Joint work with Romain Blanchard.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.
New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.

New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.

New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.
- Give the equivalence with several alternative notions of no-arbitrage previously used for utility maximisation.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.
New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.
- Give the equivalence with several alternative notions of no-arbitrage previously used for utility maximisation.
- Simple proof for FTAP.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.
New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.
- Give the equivalence with several alternative notions of no-arbitrage previously used for utility maximisation.
- Simple proof for FTAP.
- New possibility for utility maximisation.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.

New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.
- Give the equivalence with several alternative notions of no-arbitrage previously used for utility maximisation.
- Simple proof for FTAP.
- New possibility for utility maximisation.
- Revisit the so-called geometric and quantitative no-arbitrage conditions.

Aim of the paper

Investors trading in a discrete-time, multi-period and multiple-prior financial market.
New characterisation of the condition of quasi-sure no-arbitrage of [Bouchard and Nutz, 2015] which has become a standard assumption.

- Better economical understanding of this assumption.
- Give the equivalence with several alternative notions of no-arbitrage previously used for utility maximisation.
- Simple proof for FTAP.
- New possibility for utility maximisation.
- Revisit the so-called geometric and quantitative no-arbitrage conditions.
- Explicit two important examples where all these concepts are illustrated.

Framework and notations

Set of priors (similar to [Bouchard and Nutz, 2015])

- Sequence $(\Omega_t)_{1\leq t\leq T}$ of Polish spaces. We denote

$$\omega^t = (\omega_1, \ldots, \omega_t) \in \Omega^t := \Omega_1 \times \cdots \times \Omega_t.$$

Framework and notations

Set of priors (similar to [Bouchard and Nutz, 2015])

- Sequence $(\Omega_t)_{1 \leq t \leq T}$ of Polish spaces. We denote

$$\omega^t = (\omega_1, \dots, \omega_t) \in \Omega^t := \Omega_1 \times \cdots \times \Omega_t.$$

- **Assumption 1 :** the one period set of priors $\mathcal{Q}_{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathfrak{P}(\Omega_{t+1})$ is a **non-empty** and **convex valued random set** s.t.

$$\text{Graph}(\mathcal{Q}_{t+1}) = \left\{(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\right\}$$

is an **analytic set** (continuous image of a Polish space).

Framework and notations

Set of priors (similar to [Bouchard and Nutz, 2015])

- Sequence $(\Omega_t)_{1 \leq t \leq T}$ of Polish spaces. We denote

$$\omega^t = (\omega_1, \dots, \omega_t) \in \Omega^t := \Omega_1 \times \cdots \times \Omega_t.$$

- **Assumption 1 :** the one period set of priors $\mathcal{Q}_{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathfrak{P}(\Omega_{t+1})$ is a **non-empty** and **convex valued random set** s.t.

$$\text{Graph}(\mathcal{Q}_{t+1}) = \left\{(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\right\}$$

is an **analytic set** (continuous image of a Polish space).

- Jankov-von Neumann Theorem allows to construct the set of all possible priors $\mathcal{Q}^T$.

$$\mathcal{Q}^T := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_T,\ Q_1 \in \mathcal{Q}_1,\ q_{s+1} \in \mathcal{S}K_{s+1},\\ q_{s+1}(\cdot, \omega^s) \in \mathcal{Q}_{s+1}(\omega^s)\ \forall \omega^s\ \ \forall\ 1 \leq s \leq T-1\},$$

where $\mathcal{S}K_{t+1}$ is the set of $\mathcal{B}_c(\Omega^t) := \bigcap_{P \in \mathfrak{P}(\Omega^t)} \mathcal{B}_P(\Omega^t)$-meas. stochastic kernel on Ω_{t+1} given Ω^t.

Framework and notations

The traded assets and strategies

- Traded assets : $S := \{S_t,\ 0 \le t \le T\}$, d-dimensional process.

Framework and notations

The traded assets and strategies

- Traded assets : $S := \{S_t,\ 0 \leq t \leq T\}$, d-dimensional process.
- **Assumption 2 :** S is Borel-adapted.

Framework and notations

The traded assets and strategies

- Traded assets : $S := \{S_t,\ 0 \leq t \leq T\}$, d-dimensional process.
- **Assumption 2 :** S is Borel-adapted.
- Trading strategies : $\phi := \{\phi_t, 1 \leq t \leq T\} \in \Phi$, universally-predictable d-dimensional process.

Framework and notations

The traded assets and strategies

- Traded assets : $S := \{S_t,\ 0 \leq t \leq T\}$, d-dimensional process.
- **Assumption 2 :** S is Borel-adapted.
- Trading strategies : $\phi := \{\phi_t, 1 \leq t \leq T\} \in \Phi$, universally-predictable d-dimensional process.
- Trading is self-financing. Riskless asset's price is constant 1.

$$V_t^{x,\phi} = x + \sum_{s=1}^{t} \phi_s \Delta S_s.$$

No-arbitrage conditions

Monoprior NA(P)

$$NA(P): \; V_T^{0,\phi} \geq 0 \; P\text{-a.s. } \text{ for some } \phi \in \Phi \; \Rightarrow V_T^{0,\phi} = 0 \; P\text{-a.s.}$$

No-arbitrage conditions

Monoprior NA(P)

$$NA(P): \; V_T^{0,\phi} \geq 0 \; P\text{-a.s. for some } \phi \in \Phi \; \Rightarrow V_T^{0,\phi} = 0 \; P\text{-a.s.}$$

Robust NA

$$NA(\mathcal{Q}^T): \; V_T^{0,\phi} \geq 0 \; \mathcal{Q}^T\text{-q.s. for some } \phi \in \Phi \Rightarrow V_T^{0,\phi} = 0 \; \mathcal{Q}^T\text{-q.s.}$$

See Bouchard and Nutz [2015]

$N \subset \Omega^T$ is called a $\mathcal{Q}^T$-polar set if $\forall P \in \mathcal{Q}^T$, $\exists A_P \in \mathcal{B}(X)$ such that $P(A_P) = 0$ and $N \subset A_P$. Holds true $\mathcal{Q}^T$-q.s. : outside a $\mathcal{Q}^T$-polar set. $\mathcal{Q}^T$-full measure set : complement of a $\mathcal{Q}^T$-polar set.

No-arbitrage conditions

- Is $NA(\mathcal{Q}^T)$ the "right" condition ?

No-arbitrage conditions

- Is $NA(\mathcal{Q}^T)$ the "right" condition ?
- Under this condition it is not even clear if there exists a model $P \in \mathcal{Q}^T$ satisfying $NA(P)$. We will prove that this is in fact possible.

No-arbitrage conditions

- Is $NA(\mathcal{Q}^T)$ the "right" condition ?
- Under this condition it is not even clear if there exists a model $P \in \mathcal{Q}^T$ satisfying $NA(P)$. We will prove that this is in fact possible.
- Nevertheless, $\mathcal{Q}^T$ might still contain some models that are not arbitrage free.

No-arbitrage conditions

- Is $NA(\mathcal{Q}^T)$ the "right" condition ?
- Under this condition it is not even clear if there exists a model $P \in \mathcal{Q}^T$ satisfying $NA(P)$. We will prove that this is in fact possible.
- Nevertheless, $\mathcal{Q}^T$ might still contain some models that are not arbitrage free.
- An agent may not be able to delta-hedge a simple vanilla option using different levels of volatility in a arbitrage free way.

No-arbitrage conditions

Instead of $NA(\mathcal{Q}^T)$ one may assume that every model is arbitrage free.

No-arbitrage conditions

Instead of $NA(\mathcal{Q}^T)$ one may assume that every model is arbitrage free.

Strong NA

$$sNA(\mathcal{Q}^T) : NA(P),\ \forall P \in \mathcal{Q}^T.$$

No-arbitrage conditions

Instead of $NA(\mathcal{Q}^T)$ one may assume that every model is arbitrage free.

Strong NA

$$sNA(\mathcal{Q}^T) : NA(P),\ \forall P \in \mathcal{Q}^T.$$

- $sNA(\mathcal{Q}^T)$ is useful to obtain tractable theorems for expected utility maximisation for unbounded function, see [Blanchard and Carassus, 2018] and [Rásonyi and Meireles-Rodrigues, 2018].

No-arbitrage conditions

Instead of $NA(\mathcal{Q}^T)$ one may assume that every model is arbitrage free.

Strong NA

$$sNA(\mathcal{Q}^T) : NA(P),\ \forall P \in \mathcal{Q}^T.$$

- $sNA(\mathcal{Q}^T)$ is useful to obtain tractable theorems for expected utility maximisation for unbounded function, see [Blanchard and Carassus, 2018] and [Rásonyi and Meireles-Rodrigues, 2018].
- This definition seems also relevant in a continuous time setting for studying the no-arbitrage characterisation, see [Biagini et al., 2015].

No-arbitrage characterisations

Weak NA

$$wNA(\mathcal{Q}^T): \ \exists P \in \mathcal{Q}^T \ \text{ s.t. } NA(P).$$

No-arbitrage characterisations

Weak NA

$$wNA(\mathcal{Q}^T): \ \exists P \in \mathcal{Q}^T \ \text{ s.t. } NA(P).$$

The contraposition of the $wNA(\mathcal{Q}^T)$ condition is that for all models $P \in \mathcal{Q}^T$, there exists a strategy ϕ_P such that $V_T^{0,\phi_P} \geq 0$ P-a.s. and $P(V_T^{0,\phi_P} > 0) > 0$.
A concrete example of a such model-dependent arbitrage is given in [Davis and Hobson, 2007].

$$
\begin{tikzcd}[column sep=large]
\boxed{wNA\left(\mathcal{Q}^T\right)} \arrow[r, shift right, "/\!/" marking] \arrow[rr, bend left=30, "/\!/" marking] & \boxed{sNA\left(\mathcal{Q}^T\right)} \arrow[l, shift right] \arrow[r, shift right] & \boxed{NA\left(\mathcal{Q}^T\right)} \arrow[l, shift right, "/\!/" marking]
\end{tikzcd}
$$

FIGURE – Simple relations between the no-arbitrage definitions.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $\mathcal{P}^T \subset \mathcal{Q}^T$ *s.t.*

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $\mathcal{P}^T \subset \mathcal{Q}^T$ *s.t.*
    1. $\mathcal{P}^T$ *and* $\mathcal{Q}^T$ *have the same polar-sets*

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $\mathcal{P}^T \subset \mathcal{Q}^T$ *s.t.*
  1. $\mathcal{P}^T$ *and* $\mathcal{Q}^T$ *have the same polar-sets*
  2. $sNA(\mathcal{P}^T)$ *holds true i.e. NA(P) for all* $P \in \mathcal{P}^T$.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $\mathcal{P}^T \subset \mathcal{Q}^T$ *s.t.*
  1. $\mathcal{P}^T$ *and* $\mathcal{Q}^T$ *have the same polar-sets*
  2. $sNA(\mathcal{P}^T)$ *holds true i.e. NA(P) for all* $P \in \mathcal{P}^T$.

Let P^* as in Theorem 11 below with the fix disintegration $P^* := P_1^* \otimes p_2^* \otimes \cdots \otimes p_T^*$. The set $\mathcal{P}^T$ is defined recursively as follows :

$$\mathcal{P}^1 := \left\{\lambda P_1^* + (1-\lambda)P,\ 0 < \lambda \leq 1,\ P \in \mathcal{Q}^1\right\},$$

$$\mathcal{P}^{t+1} := \Big\{P_t \otimes \left(\lambda p_{t+1}^* + (1-\lambda)q_{t+1}\right),\ 0 < \lambda \leq 1,$$
$$P_t \in \mathcal{P}^t,\ q_{t+1}(\cdot,\omega^t) \in \mathcal{Q}_{t+1}(\omega^t)\ \ \forall \omega^t \in \Omega^t\Big\}.$$

Applications

Equivalence between the $NA(\mathcal{Q}^T)$ condition and the no-arbitrage condition introduced by [Bartl et al., 2019] which studies the problem of robust maximisation of expected utility using medial limits.

Applications

Equivalence between the $NA(\mathcal{Q}^T)$ condition and the no-arbitrage condition introduced by [Bartl et al., 2019] which studies the problem of robust maximisation of expected utility using medial limits.

Corollary

Assume that Assumptions 1. and 2. hold true. The following conditions are equivalent

- *The* $NA(\mathcal{Q}^T)$ *condition holds true.*
- *For all* $Q \in \mathcal{Q}^T$, *there exists some* $P \in \mathcal{P}^T$ *such that* $Q \ll P$ *and such that* $NA(P)$ *holds true.*
- *For all* $Q \in \mathcal{Q}^T$, *there exists some* $P \in \mathcal{Q}^T$ *such that* $Q \ll P$ *and such that* $NA(P)$ *holds true.*

Applications

Robust FTAP of [Bouchard and Nutz, 2015].

$$\mathcal{R}^T := \{P \in \mathfrak{P}(\Omega^T),\ \exists Q^{'} \in \mathcal{Q}^T, P \ll Q^{'} \text{ and } P \text{ is a martingale measure}\}.$$
$$\mathcal{K}^T := \{P \in \mathfrak{P}(\Omega^T),\ \exists Q^{'} \in \mathcal{P}^T, P \sim Q^{'} \text{ and } P \text{ is a martingale measure}\}.$$

Applications

Robust FTAP of [Bouchard and Nutz, 2015].

$$\mathcal{R}^T := \{P \in \mathfrak{P}(\Omega^T),\ \exists Q' \in \mathcal{Q}^T, P \ll Q' \text{ and } P \text{ is a martingale measure}\}.$$
$$\mathcal{K}^T := \{P \in \mathfrak{P}(\Omega^T),\ \exists Q' \in \mathcal{P}^T, P \sim Q' \text{ and } P \text{ is a martingale measure}\}.$$

Corollary

Assume that Assumptions 1. and 2. hold true. The following conditions are equivalent

- *The* $NA(\mathcal{Q}^T)$ *condition holds true.*
- *For all* $Q \in \mathcal{Q}^T$*, there exists some* $P \in \mathcal{K}^T$ *such that* $Q \ll P$.
- *For all* $Q \in \mathcal{Q}^T$*, there exists some* $P \in \mathcal{R}^T$ *such that* $Q \ll P$.

Applications

Random Utility

$U : \Omega^T \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

- for every $x \in \mathbb{R}$, $U(\cdot, x) : \Omega^T \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{B}(\Omega^T)$-measurable,
- for all $\omega^T \in \Omega^T$, $U(\omega^T, \cdot) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and concave on $(0, \infty)$
- $U(\cdot, x) = -\infty$, for all $x < 0$.

Applications

Random Utility

$U: \ \Omega^T \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

- for every $x \in \mathbb{R}$, $U(\cdot, x): \ \Omega^T \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{B}(\Omega^T)$-measurable,
- for all $\omega^T \in \Omega^T$, $U\left(\omega^T, \cdot\right): \ \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and concave on $(0, \infty)$
- $U(\cdot, x) = -\infty$, for all $x < 0$.

Robust portfolio problem with initial wealth x

$$u(x) := \sup_{\phi \in \Phi(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)). \tag{1}$$

where $\Phi(x, U, \mathcal{Q}^T)$ is the set of all strategies, s.t $V_T^{x,\phi}(\cdot) \geq 0$ $\mathcal{Q}^T$-q.s. and $E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $E_P U^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ for all $P \in \mathcal{Q}^T$.

Applications

Assumption 3 : We have that $U^+(\cdot, 1), U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ and $\Delta S_t, 1/\alpha_t^P \in \mathcal{W}_t$ for all $1 \leq t \leq T$ and $P \in \mathcal{P}^t$, where

$$\mathcal{W}_t := \bigcap_{r>0} \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}(\Omega^t)\text{-measurable},\ \sup_{P \in \mathcal{Q}^t} E_P|X|^r < \infty \right\}.$$

Applications

Assumption 3 : We have that $U^+(\cdot,1), U^-(\cdot,\frac{1}{4}) \in \mathcal{W}_T$ and $\Delta S_t, 1/\alpha_t^P \in \mathcal{W}_t$ for all $1 \leq t \leq T$ and $P \in \mathcal{P}^t$, where

$$\mathcal{W}_t := \bigcap_{r>0} \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}(\Omega^t)\text{-measurable},\ \sup_{P \in \mathcal{Q}^t} E_P|X|^r < \infty \right\}.$$

Corollary

Assume that the $NA(\mathcal{Q}^T)$ condition and Assumptions 1. and 2. hold true. Furthermore, assume that U is either bounded from above or that Assumption 3. holds true. Then for all $x \geq 0$.

$$u(x) = u^{\mathcal{P}}(x) := \sup_{\phi \in \Phi(x,U,\mathcal{P}^T)} \inf_{P \in \mathcal{P}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)).$$

Local NA

First part of [Bouchard and Nutz, 2015, Theorem 4.5]

Theorem

Assume that Assumptions 1 and 2 hold true. Then the following statements are equivalent :
1. The $NA(\mathcal{Q}^T)$ condition holds true.
2. For all $0 \leq t \leq T-1$, there exists a $\mathcal{Q}^t$-full measure set $\Omega^t_{NA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{NA}$,

$$h\Delta S_{t+1}(\omega^t, \cdot) \geq 0 \ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.} \Rightarrow h\Delta S_{t+1}(\omega^t, \cdot) = 0 \ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.}$$

Measurability of the supports

Lemma

Let $P \in \mathcal{Q}^T$ with a fixed disintegration $P := Q_1 \otimes q_2 \otimes \cdots \otimes q_T$. Under Assumptions 1 and 2, the following supports of the conditional distribution of $\Delta S_{t+1}(\omega^t, \cdot)$

$$D^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \; P_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A\right) = 1, \; \forall\, P_{t+1} \in \mathcal{Q}_{t+1}(\omega^t) \right\}$$

$$D_P^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \; q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A, \omega^t\right) = 1 \right\},$$

are non-empty, closed valued random set with graphs in $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.

Geometric NA

Geometric view in the spirit of [Jacod and Shiryaev, 1998, Theorem 3g)].
Recall that $\text{Ri}(C) = \{y \in C, \exists \varepsilon > 0, \text{ Aff}(C) \cap B(y, \varepsilon) \subset C\}$.

Definition

The geometric no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{gNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{gNA}$, $0 \in \text{Ri}\left(\text{Conv}(D^{t+1})\right)(\omega^t)$.
In this case for all $\omega^t \in \Omega^t_{gNA}$, there exists $\varepsilon_t(\omega^t) > 0$ such that

$$B(0, \varepsilon_t(\omega^t)) \cap \text{Aff}\left(D^{t+1}\right)(\omega^t) \subset \text{Conv}\left(D^{t+1}\right)(\omega^t).$$

Geometric NA

Geometric view in the spirit of [Jacod and Shiryaev, 1998, Theorem 3g)]. Recall that $\mathrm{Ri}(C) = \{y \in C, \exists \varepsilon > 0, \ \mathrm{Aff}(C) \cap B(y, \varepsilon) \subset C\}$.

Definition

The geometric no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{gNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{gNA}$, $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D^{t+1})\right)(\omega^t)$.
In this case for all $\omega^t \in \Omega^t_{gNA}$, there exists $\varepsilon_t(\omega^t) > 0$ such that

$$B(0, \varepsilon_t(\omega^t)) \cap \mathrm{Aff}\left(D^{t+1}\right)(\omega^t) \subset \mathrm{Conv}\left(D^{t+1}\right)(\omega^t).$$

- The geometric (local) no-arbitrage condition is indeed practical : it allows to check whether the (global) NA($\mathcal{Q}^T$) condition holds true or not.

Geometric NA

Geometric view in the spirit of [Jacod and Shiryaev, 1998, Theorem 3g)].
Recall that $\mathrm{Ri}(C) = \{y \in C, \exists\, \varepsilon > 0,\ \mathrm{Aff}(C) \cap B(y, \varepsilon) \subset C\}$.

Definition

The geometric no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{gNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{gNA}$, $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D^{t+1})\right)(\omega^t)$.
In this case for all $\omega^t \in \Omega^t_{gNA}$, there exists $\varepsilon_t(\omega^t) > 0$ such that

$$B(0, \varepsilon_t(\omega^t)) \cap \mathrm{Aff}\left(D^{t+1}\right)(\omega^t) \subset \mathrm{Conv}\left(D^{t+1}\right)(\omega^t).$$

- The geometric (local) no-arbitrage condition is indeed practical : it allows to check whether the (global) $\mathrm{NA}(\mathcal{Q}^T)$ condition holds true or not.
- As $\mathcal{Q}_{t+1}$ and ΔS_{t+1} are given one gets $\mathrm{Ri}\left(\mathrm{Conv}(D^{t+1})\right)(\cdot)$ and it is easy to check whether 0 is in it or not.

Quantitative NA

Quantitative view the spirit of [Rásonyi and Stettner, 2005, Proposition 3.3]

Definition

The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{qNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{qNA}$, there exists $\beta_t(\omega^t), \kappa_t(\omega^t) \in (0,1]$ such that for all $h \in \text{Aff}\left(D^{t+1}\right)(\omega^t)$, $h \neq 0$ there exists $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t,\cdot) < -\beta_t(\omega^t)\right) \geq \kappa_t(\omega^t).$$

Quantitative NA

Quantitative view the spirit of [Rásonyi and Stettner, 2005, Proposition 3.3]

Definition

The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{qNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{qNA}$, there exists $\beta_t(\omega^t), \kappa_t(\omega^t) \in (0,1]$ such that for all $h \in \text{Aff}\left(D^{t+1}\right)(\omega^t)$, $h \neq 0$ there exists $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, \cdot) < -\beta_t(\omega^t)\right) \geq \kappa_t(\omega^t).$$

- One risky asset and one period : there exists a prior for which the price of the risky asset increases enough and an other one for which it decreases, $P^{\pm}\left(\mp\Delta S(\cdot) < -\alpha\right) > \alpha$ where $\alpha > 0$.

Quantitative NA

Quantitative view the spirit of [Rásonyi and Stettner, 2005, Proposition 3.3]

Definition

The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{qNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{qNA}$, there exists $\beta_t(\omega^t), \kappa_t(\omega^t) \in (0,1]$ such that for all $h \in \text{Aff}\left(D^{t+1}\right)(\omega^t)$, $h \neq 0$ there exists $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t,\cdot) < -\beta_t(\omega^t)\right) \geq \kappa_t(\omega^t).$$

- One risky asset and one period : there exists a prior for which the price of the risky asset increases enough and an other one for which it decreases, $P^{\pm}\left(\mp\Delta S(\cdot) < -\alpha\right) > \alpha$ where $\alpha > 0$.
- The probability measure depends of the strategy.

Quantitative NA

- $\beta_t(\omega^t)$ **provides information on** $D^{t+1}(\omega^t)$ **while** $\kappa_t(\omega^t)$ **provides information on** $\mathcal{Q}_{t+1}(\omega^t)$**.**

Quantitative NA

- **$\beta_t(\omega^t)$ provides information on $D^{t+1}(\omega^t)$ while $\kappa_t(\omega^t)$ provides information on $\mathcal{Q}_{t+1}(\omega^t)$.**
- Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.

Quantitative NA

- $\beta_t(\omega^t)$ **provides information on** $D^{t+1}(\omega^t)$ **while** $\kappa_t(\omega^t)$ **provides information on** $\mathcal{Q}_{t+1}(\omega^t)$**.**
- Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.
- When $\mathrm{Dom}(U) = (0, \infty)$ it provides natural bounds for the one step strategies or for $U(V_T^{x,\Phi})$, see [Blanchard and Carassus, 2018].

Quantitative NA

- $\beta_t(\omega^t)$ **provides information on** $D^{t+1}(\omega^t)$ **while** $\kappa_t(\omega^t)$ **provides information on** $\mathcal{Q}_{t+1}(\omega^t)$**.**
- Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.
- When $\mathrm{Dom}(U) = (0, \infty)$ it provides natural bounds for the one step strategies or for $U(V_T^{x,\Phi})$, see [Blanchard and Carassus, 2018].
- Used to prove the existence of the optimal strategy but could also be used to compute it numerically.

Quantitative NA

- $\beta_t(\omega^t)$ **provides information on** $D^{t+1}(\omega^t)$ **while** $\kappa_t(\omega^t)$ **provides information on** $\mathcal{Q}_{t+1}(\omega^t)$**.**
- Quantitative (local) no-arbitrage condition is precious for solving the problem of maximisation of expected utility.
- When $\mathrm{Dom}(U) = (0, \infty)$ it provides natural bounds for the one step strategies or for $U(V_T^{x,\Phi})$, see [Blanchard and Carassus, 2018].
- Used to prove the existence of the optimal strategy but could also be used to compute it numerically.
- Explicit values for β_t and κ_t are given.

Theorem

Assume that Assumptions 1 and 2 hold true. Then the $NA(\mathcal{Q}^T)$ condition, the geometric no-arbitrage and the quantitative no-arbitrage are equivalent and one can choose for all $0 \leq t \leq T-1$

$$\Omega^t_{NA} = \Omega^t_{qNA} = \Omega^t_{gNA}$$

and $\beta_t = \varepsilon_t/2$.

Theorem

Assume that Assumptions 1 and 2 hold true. Then the $NA(\mathcal{Q}^T)$ *condition, the geometric no-arbitrage and the quantitative no-arbitrage are equivalent and one can choose for all* $0 \leq t \leq T-1$

$$\Omega^t_{NA} = \Omega^t_{qNA} = \Omega^t_{gNA}$$

and $\beta_t = \varepsilon_t/2$.

Proposition

Assume that Assumptions 1 and 2 hold true. Under one of the no-arbitrage conditions one can choose an universally measurable version of ε_t *and* β_t.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1$, $P_1(\Delta S_1^1 > 0) > 0$.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1$, $P_1(\Delta S_1^1 > 0) > 0$.
- Let P_2 s.t. $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm\Delta S_1^2 > 0) > 0$.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1,\ P_1(\Delta S_1^1 > 0) > 0$.
- Let P_2 s.t. $P_2(\Delta S_1^1 = 0) = 1,\ P_2(\pm\Delta S_1^2 > 0) > 0$.
- $\mathcal{Q} = \{\lambda P_1 + (1-\lambda)P_2,\ 0 < \lambda \leq 1\}$.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1,\ P_1(\Delta S_1^1 > 0) > 0$.
- Let P_2 s.t. $P_2(\Delta S_1^1 = 0) = 1,\ P_2(\pm\Delta S_1^2 > 0) > 0$.
- $\mathcal{Q} = \{\lambda P_1 + (1-\lambda)P_2,\ 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(\mathcal{Q})$ hold true.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1$, $P_1(\Delta S_1^1 > 0) > 0$.
- Let P_2 s.t. $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm\Delta S_1^2 > 0) > 0$.
- $\mathcal{Q} = \{\lambda P_1 + (1-\lambda) P_2,\ 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(\mathcal{Q})$ hold true.
- $NA(\mathcal{Q})$ condition does not hold true : Let $h = (1, 0)$. Then $h\Delta S_1 \geq 0$ $\mathcal{Q}$-q.s. but $P_1(h\Delta S_1 > 0) > 0$.

Probability measure P^*

- $wNA(\mathcal{Q}^T)$ does not imply $NA(\mathcal{Q}^T)$ condition.
- One period model with two risky assets $S_0^i = 0$ and $S_1^i : \Omega \to \mathbb{R}$.
- Let P_1 s.t. $P_1(\Delta S_1^1 \geq 0) = 1$, $P_1(\Delta S_1^1 > 0) > 0$.
- Let P_2 s.t. $P_2(\Delta S_1^1 = 0) = 1$, $P_2(\pm\Delta S_1^2 > 0) > 0$.
- $\mathcal{Q} = \{\lambda P_1 + (1-\lambda)P_2,\ 0 < \lambda \leq 1\}$.
- $NA(P_2)$ and the $wNA(\mathcal{Q})$ hold true.
- $NA(\mathcal{Q})$ condition does not hold true : Let $h = (1, 0)$. Then $h\Delta S_1 \geq 0$ $\mathcal{Q}$-q.s. but $P_1(h\Delta S_1 > 0) > 0$.
- Note that Aff$(D) = \mathbb{R}^2$ and Aff$\left(D_{P_2}\right) = \{0\} \times \mathbb{R}$.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
    - $\mathit{Aff}\left(D^{t+1}_{P^*}\right)(\omega^t) = \mathit{Aff}\left(D^{t+1}\right)(\omega^t)$

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - $Aff\left(D^{t+1}_{P^*}\right)(\omega^t) = Aff\left(D^{t+1}\right)(\omega^t)$
  - $0 \in Ri\left(Conv(D^{t+1}_{P^*})\right)(\omega^t)$.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - *Aff* $\left(D^{t+1}_{P^*}\right)(\omega^t) = \textit{Aff}\left(D^{t+1}\right)(\omega^t)$
  - $0 \in \textit{Ri}\left(\textit{Conv}(D^{t+1}_{P^*})\right)(\omega^t)$.

- $NA(P^*)$ condition holds true and even more.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - $\mathit{Aff}\left(D^{t+1}_{P^*}\right)(\omega^t) = \mathit{Aff}\left(D^{t+1}\right)(\omega^t)$
  - $0 \in \mathit{Ri}\left(\mathit{Conv}(D^{t+1}_{P^*})\right)(\omega^t)$.

- $NA(P^*)$ condition holds true and even more.
- The condition $\mathrm{Aff}\left(D^{t+1}_{P^*}\right)(\cdot) = \mathrm{Aff}\left(D^{t+1}\right)(\cdot)$ $\mathcal{Q}^t$-q.s. is necessary (see the preceding counterexample).

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - *Aff* $\left(D^{t+1}_{P^*}\right)(\omega^t) =$ *Aff* $\left(D^{t+1}\right)(\omega^t)$
  - $0 \in$ *Ri* $\left(\text{\textit{Conv}}(D^{t+1}_{P^*})\right)(\omega^t)$.

- $NA(P^*)$ condition holds true and even more.
- The condition Aff $\left(D^{t+1}_{P^*}\right)(\cdot) =$ Aff $\left(D^{t+1}\right)(\cdot)$ $\mathcal{Q}^t$-q.s. is necessary (see the preceding counterexample).
- Other counterexample if $0 \in$ Ri $\left(\text{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ P^*_t-p.s. instead of $0 \in$ Ri $\left(\text{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ $\mathcal{Q}^t$-q.s.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - $\mathit{Aff}\left(D^{t+1}_{P^*}\right)(\omega^t) = \mathit{Aff}\left(D^{t+1}\right)(\omega^t)$
  - $0 \in \mathit{Ri}\left(\mathit{Conv}(D^{t+1}_{P^*})\right)(\omega^t)$.

- $NA(P^*)$ condition holds true and even more.
- The condition $\mathrm{Aff}\left(D^{t+1}_{P^*}\right)(\cdot) = \mathrm{Aff}\left(D^{t+1}\right)(\cdot)$ $\mathcal{Q}^t$-q.s. is necessary (see the preceding counterexample).
- Other counterexample if $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ P^*_t-p.s. instead of $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ $\mathcal{Q}^t$-q.s.
- P^* was used to build $\mathcal{P}^T$.

Theorem

Assume that Assumptions 1. and 2. hold true. TFAE

- $NA(\mathcal{Q}^T)$ *holds true.*
- *There exists some* $P^* \in \mathcal{Q}^T$ *such that for all* $0 \leq t \leq T-1$, $\omega^t \in \Omega^t_{NA}$
  - *Aff* $\left(D^{t+1}_{P^*}\right)(\omega^t) =$ *Aff* $\left(D^{t+1}\right)(\omega^t)$
  - $0 \in$ *Ri* $\left(\textit{Conv}(D^{t+1}_{P^*})\right)(\omega^t)$.

- $NA(P^*)$ condition holds true and even more.
- The condition Aff $\left(D^{t+1}_{P^*}\right)(\cdot) =$ Aff $\left(D^{t+1}\right)(\cdot)$ $\mathcal{Q}^t$-q.s. is necessary (see the preceding counterexample).
- Other counterexample if $0 \in$ Ri $\left(\text{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ P^*_t-p.s. instead of $0 \in$ Ri $\left(\text{Conv}(D^{t+1}_{P^*})\right)(\cdot)$ $\mathcal{Q}^t$-q.s.
- P^* was used to build $\mathcal{P}^T$.
- The probability measure P^* of Theorem 11 is not unique.

- Complement [Obłój and Wiesel, 2018, Theorem 3.1] which makes the link with the quasi-sure setting.

- Complement [Obłój and Wiesel, 2018, Theorem 3.1] which makes the link with the quasi-sure setting.
- The existence of P^* show that $NA(\mathcal{Q}^T)$ implies that (an adaptation of) [Rásonyi and Meireles-Rodrigues, 2018, Assumption 2.1] and thus [Rásonyi and Meireles-Rodrigues, 2018, Theorem 3.7] which shows the existence in the problem of maximisation of expected utility for bounded function defined on the whole real line works under $NA(\mathcal{Q}^T)$.

- Complement [Obłój and Wiesel, 2018, Theorem 3.1] which makes the link with the quasi-sure setting.
- The existence of P^* show that $NA(\mathcal{Q}^T)$ implies that (an adaptation of) [Rásonyi and Meireles-Rodrigues, 2018, Assumption 2.1] and thus [Rásonyi and Meireles-Rodrigues, 2018, Theorem 3.7] which shows the existence in the problem of maximisation of expected utility for bounded function defined on the whole real line works under $NA(\mathcal{Q}^T)$.
- One can choose P^* as common probability measure in the quantitative definition of NA.

- Complement [Obłój and Wiesel, 2018, Theorem 3.1] which makes the link with the quasi-sure setting.
- The existence of P^* show that $NA(\mathcal{Q}^T)$ implies that (an adaptation of) [Rásonyi and Meireles-Rodrigues, 2018, Assumption 2.1] and thus [Rásonyi and Meireles-Rodrigues, 2018, Theorem 3.7] which shows the existence in the problem of maximisation of expected utility for bounded function defined on the whole real line works under $NA(\mathcal{Q}^T)$.
- One can choose P^* as common probability measure in the quantitative definition of NA.
- Allows to find universally measurable version of κ_t.

Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure $\widehat{P} \in \mathcal{Q}^T$. Then

- *The $NA(\widehat{P})$ and the $NA(\mathcal{Q}^T)$ conditions are equivalent.*

Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure $\widehat{P} \in \mathcal{Q}^T$. Then

- *The $NA(\widehat{P})$ and the $NA(\mathcal{Q}^T)$ conditions are equivalent.*
- *One can choose $P^* = \widehat{P}$ in $\mathcal{P}^T$.*

Proposition

Assume that Assumptions 1. and 2. hold true. Assume furthermore that there exists some dominating measure $\widehat{P} \in \mathcal{Q}^T$. Then

- *The $NA(\widehat{P})$ and the $NA(\mathcal{Q}^T)$ conditions are equivalent.*
- *One can choose $P^* = \widehat{P}$ in $\mathcal{P}^T$.*

Proposition

Assume that Assumption 2. holds true and that there exists

1. *some $\widetilde{P} \in \mathcal{Q}^T$*
2. *some $0 \leq \widetilde{t} \leq T-1$ and some $\Omega_N^{\widetilde{t}} \in \mathcal{B}_c(\Omega^{\widetilde{t}})$ such that*

- $\widetilde{P}^{\widetilde{t}}(\Omega_N^{\widetilde{t}}) > 0$
- $\mathcal{Q}_{\widetilde{t}+1}(\omega^{\widetilde{t}})$ *is* **not dominated** *for all $\omega^{\widetilde{t}} \in \Omega_N^{\widetilde{t}}$.*

Then $\mathcal{Q}^T$ is **not dominated***.*

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\begin{aligned}\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t),\\ u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},\end{aligned}$$

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\begin{aligned}\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t),\\ u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},\end{aligned}$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t),\\ u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0,1]$ and $p_t(\omega^t) < 1,\ P_t(\omega^t) > 0$

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\begin{aligned} \mathcal{B}_{t+1}(\omega^t) := \{ & p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t), \\ & u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\}, \end{aligned}$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0,1]$ and $p_t(\omega^t) < 1,\ P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\begin{aligned}\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t),\\ u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},\end{aligned}$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0,1]$ and $p_t(\omega^t) < 1,\ P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.
- Then

$$\mathcal{Q}_{t+1}(\omega^t) := \text{Conv}\left(\left\{Q \in \mathfrak{P}(\Omega_{t+1}),\ Q\left(Y_{t+1} \in \cdot\right) \in \mathcal{B}_{t+1}(\omega^t)\right\}\right),$$

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ p_t(\omega^t) \leq p \leq P_t(\omega^t),\\ u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0,1]$ and $p_t(\omega^t) < 1,\ P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.
- Then

$$\mathcal{Q}_{t+1}(\omega^t) := \text{Conv}\left(\{Q \in \mathfrak{P}(\Omega_{t+1}),\ Q\,(Y_{t+1} \in \cdot) \in \mathcal{B}_{t+1}(\omega^t)\}\right),$$

- Usual binomial model corresponds : $p_t = P_t = p$, $u_t = U_t = u$ and $d_t = D_T = d$ where $0 < p < 1,\ d < 1 < u$.

- Suppose that $T \geq 1$, $d = 1$ and $\Omega_t = \mathbb{R}$ for all $1 \leq t \leq T$.
- $S_0 = 1$ and $S_{t+1} = S_t Y_{t+1}$ where Y_{t+1} Borel-measurable r.v. s.t. $Y_{t+1}(\Omega_{t+1}) = (0, \infty)$.
- Assumption 1. is verified. Let

$$\begin{aligned}\mathcal{B}_{t+1}(\omega^t) := \{p\delta_u + (1-p)\delta_d,\ & p_t(\omega^t) \leq p \leq P_t(\omega^t), \\ & u_t(\omega^t) \leq u \leq U_t(\omega^t),\ d_t(\omega^t) \leq d \leq D_t(\omega^t)\},\end{aligned}$$

- where $p_t, P_t, u_t, U_t, d_t, D_t$ are Borel-measurable r.v. s.t.
- $p_t(\omega^t), P_t(\omega^t) \in [0, 1]$ and $p_t(\omega^t) < 1,\ P_t(\omega^t) > 0$
- $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$.
- Then

$$\mathcal{Q}_{t+1}(\omega^t) := \text{Conv}\left(\left\{Q \in \mathfrak{P}(\Omega_{t+1}),\ Q\left(Y_{t+1} \in \cdot\right) \in \mathcal{B}_{t+1}(\omega^t)\right\}\right),$$

- Usual binomial model corresponds : $p_t = P_t = p$, $u_t = U_t = u$ and $d_t = D_T = d$ where $0 < p < 1,\ d < 1 < u$.
- Assumption 2. holds true.

- $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$

- $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
- $\text{Conv}\left(D^{t+1}\right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.

- $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
- Conv $\left(D^{t+1}\right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.
- $NA(\mathcal{Q}^T)$ holds true : $0 \in$ Ri $\left(\text{Conv}\left(D^{t+1}\right)\right)(\omega^t)$ for all $\omega^t \in \Omega^t$.

- $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
- Conv $\left(D^{t+1}\right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.
- $NA(\mathcal{Q}^T)$ holds true : $0 \in \mathrm{Ri}\left(\mathrm{Conv}\left(D^{t+1}\right)\right)(\omega^t)$ for all $\omega^t \in \Omega^t$.
- If for instance $u_t(\omega^t) < 1$ for all ω^t, $\exists a_t(\omega^t) \in [u_t(\omega^t), 1)$. Let for $\pi_t(\omega^t) \in [p_t(\omega^t), P_t(\omega^t)]$

$$
\begin{aligned}
q_{t+1}(\Delta S_{t+1} \in \cdot, \omega^t) &= \pi_t(\omega^t)\delta_{a_t(\omega^t)}(\cdot) + \left(1 - \pi_t(\omega^t)\right)\delta_{d_t(\omega^t)}(\cdot) \\
Q &= Q_1 \otimes \cdots \otimes q_t \in \mathcal{Q}^T \\
\mathrm{Conv}\left(D_Q^{t+1}\right)(\omega^t) &= \left[S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(a_t(\omega^t) - 1)\right]
\end{aligned}
$$

- $S_{t+1} - S_t = S_t(Y_{t+1} - 1)$ and $0 < d_t(\omega^t) < 1 < U_t(\omega^t)$
- Conv $\left(D^{t+1}\right)(\omega^t) = [S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(U_t(\omega^t) - 1)]$.
- $NA(\mathcal{Q}^T)$ holds true : $0 \in \text{Ri}\left(\text{Conv}\left(D^{t+1}\right)\right)(\omega^t)$ for all $\omega^t \in \Omega^t$.
- If for instance $u_t(\omega^t) < 1$ for all ω^t, $\exists a_t(\omega^t) \in [u_t(\omega^t), 1)$. Let for $\pi_t(\omega^t) \in [p_t(\omega^t), P_t(\omega^t)]$

$$\begin{aligned} q_{t+1}(\Delta S_{t+1} \in \cdot, \omega^t) &= \pi_t(\omega^t)\delta_{a_t(\omega^t)}(\cdot) + \left(1 - \pi_t(\omega^t)\right)\delta_{d_t(\omega^t)}(\cdot) \\ Q &= Q_1 \otimes \cdots \otimes q_t \in \mathcal{Q}^T \\ \text{Conv}\left(D_Q^{t+1}\right)(\omega^t) &= \left[S_t(\omega^t)(d_t(\omega^t) - 1), S_t(\omega^t)(a_t(\omega^t) - 1)\right] \end{aligned}$$

- $0 \notin \text{Ri}\left(\text{Conv}\left(D_Q^{t+1}\right)\right)(\omega^t)$ for all $\omega^t \in \Omega^t$ and both the $NA(Q)$ and $sNA(\mathcal{Q}^T)$ conditions fail.

- Explicit expressions for ε_t, β_t and κ_t

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t)-1}{2}, \frac{1-d_t(\omega^t)}{2}\right) > 0,$$

$$\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t)+P_t(\omega^t)}{2}, 1 - \frac{p_t(\omega^t)+P_t(\omega^t)}{2}\right) > 0.$$

- Explicit expressions for ε_t, β_t and κ_t

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t)-1}{2}, \frac{1-d_t(\omega^t)}{2}\right) > 0,$$

$$\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t)+P_t(\omega^t)}{2}, 1-\frac{p_t(\omega^t)+P_t(\omega^t)}{2}\right) > 0.$$

- $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.

- Explicit expressions for ε_t, β_t and κ_t

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t)-1}{2}, \frac{1-d_t(\omega^t)}{2}\right) > 0,$$

$$\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t)+P_t(\omega^t)}{2}, 1-\frac{p_t(\omega^t)+P_t(\omega^t)}{2}\right) > 0.$$

- $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.
- The (Borel) measurability of ε_t, β_t and κ_t are clear.

- Explicit expressions for ε_t, β_t and κ_t

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t)-1}{2}, \frac{1-d_t(\omega^t)}{2}\right) > 0,$$

$$\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t)+P_t(\omega^t)}{2}, 1-\frac{p_t(\omega^t)+P_t(\omega^t)}{2}\right) > 0.$$

- $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.
- The (Borel) measurability of ε_t, β_t and κ_t are clear.
- Let $\bar{\pi}_t(\omega^t) := \frac{p_t(\omega^t)+P_t(\omega^t)}{2} \in (0,1)$ and $a^{\pm}, b^{\pm}$ be chosen such that

$$a_t^+(\omega^t) := U_t(\omega^t) > 1, \quad b_t^+(\omega^t) := \min\left(D_t(\omega^t), \frac{d_t(\omega^t)+1}{2}\right) < 1,$$

$$a_t^-(\omega^t) := \max\left(u_t(\omega^t), \frac{U_t(\omega^t)+1}{2}\right) > 1, \quad b_t^-(\omega^t) := d_t(\omega^t) < 1,$$

$$r_{t+1}^{\pm}(\cdot,\omega^t) := \bar{\pi}_t(\omega^t)\delta_{a_t^{\pm}(\omega^t)}(\cdot) + (1-\bar{\pi}_t(\omega^t))\delta_{b_t^{\pm}(\omega^t)}(\cdot) \in \mathcal{B}_{t+1}(\omega^t),$$

$$r_{t+1}^*(\cdot,\omega^t) := \frac{1}{2}\left(r_{t+1}^+(\cdot,\omega^t) + r_{t+1}^-(\cdot,\omega^t)\right) \in \mathcal{B}_{t+1}(\omega^t),$$

$$p_{t+1}^*(Y_{t+1} \in \cdot,\omega^t) := r_{t+1}^*(\cdot,\omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$$

- Explicit expressions for ε_t, β_t and κ_t

$$\frac{\varepsilon_t(\omega^t)}{2} = \beta_t(\omega^t) = \frac{S_t(\omega^t)}{N} \min\left(\frac{U_t(\omega^t)-1}{2}, \frac{1-d_t(\omega^t)}{2}\right) > 0,$$

$$\kappa_t(\omega^t) = \frac{1}{M} \min\left(\frac{p_t(\omega^t)+P_t(\omega^t)}{2}, 1-\frac{p_t(\omega^t)+P_t(\omega^t)}{2}\right) > 0.$$

- $N > 1$ and $M \geq 1$ are fixed and allows to get sharper bounds.
- The (Borel) measurability of ε_t, β_t and κ_t are clear.
- Let $\bar{\pi}_t(\omega^t) := \frac{p_t(\omega^t)+P_t(\omega^t)}{2} \in (0,1)$ and $a^{\pm}, b^{\pm}$ be chosen such that

$$a_t^+(\omega^t) := U_t(\omega^t) > 1, \quad b_t^+(\omega^t) := \min\left(D_t(\omega^t), \frac{d_t(\omega^t)+1}{2}\right) < 1,$$

$$a_t^-(\omega^t) := \max\left(u_t(\omega^t), \frac{U_t(\omega^t)+1}{2}\right) > 1, \quad b_t^-(\omega^t) := d_t(\omega^t) < 1,$$

$$r_{t+1}^{\pm}(\cdot,\omega^t) := \bar{\pi}_t(\omega^t)\delta_{a_t^{\pm}(\omega^t)}(\cdot) + (1-\bar{\pi}_t(\omega^t))\delta_{b_t^{\pm}(\omega^t)}(\cdot) \in \mathcal{B}_{t+1}(\omega^t),$$

$$r_{t+1}^{*}(\cdot,\omega^t) := \frac{1}{2}\left(r_{t+1}^{+}(\cdot,\omega^t) + r_{t+1}^{-}(\cdot,\omega^t)\right) \in \mathcal{B}_{t+1}(\omega^t),$$

$$p_{t+1}^{*}(Y_{t+1} \in \cdot, \omega^t) := r_{t+1}^{*}(\cdot,\omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$$

$$p_{t+1}^{*}\left(\pm\Delta S_{t+1}(\omega^t,\cdot) < -\beta_t(\omega^t), \omega^t\right) \geq \kappa_t(\omega^t).$$

- Choose

- Choose
- $P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in \mathcal{Q}^T$. $0 \in \mathsf{Ri}\left(\mathsf{Conv}(D_{P^*}^{t+1})\right)(\omega^t)$ and that $\mathsf{Aff}\left(D_{P^*}^{t+1}\right)(\omega^t) = \mathsf{Aff}\left(D^{t+1}\right)(\omega^t)$ for all ω^t.

- Choose
- $P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in \mathcal{Q}^T$. $0 \in \mathsf{Ri}\left(\mathsf{Conv}(D_{P^*}^{t+1})\right)(\omega^t)$ and that $\mathsf{Aff}\left(D_{P^*}^{t+1}\right)(\omega^t) = \mathsf{Aff}\left(D^{t+1}\right)(\omega^t)$ for all ω^t.
- Note that P^* is not unique.

- Choose
- $P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in \mathcal{Q}^T$. $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D_{P^*}^{t+1})\right)(\omega^t)$ and that $\mathrm{Aff}\left(D_{P^*}^{t+1}\right)(\omega^t) = \mathrm{Aff}\left(D^{t+1}\right)(\omega^t)$ for all ω^t.
- Note that P^* is not unique.
- Finally, if for some $0 \leq t \leq T-1$, $\omega^t \in \Omega^t$, $u_t(\omega^t) < U_t(\omega^t)$ or $d_t(\omega^t) < D_t(\omega^t)$ the set $\mathcal{Q}_{t+1}(\omega^t)$ is non-dominated and $\mathcal{Q}^T$ is also non-dominated.

- Choose
- $P^* := P_0^* \otimes \cdots \otimes p_{t+1}^* \otimes \cdots p_T^* \in \mathcal{Q}^T$. $0 \in \mathrm{Ri}\left(\mathrm{Conv}(D_{P^*}^{t+1})\right)(\omega^t)$ and that $\mathrm{Aff}\left(D_{P^*}^{t+1}\right)(\omega^t) = \mathrm{Aff}\left(D^{t+1}\right)(\omega^t)$ for all ω^t.
- Note that P^* is not unique.
- Finally, if for some $0 \leq t \leq T-1$, $\omega^t \in \Omega^t$, $u_t(\omega^t) < U_t(\omega^t)$ or $d_t(\omega^t) < D_t(\omega^t)$ the set $\mathcal{Q}_{t+1}(\omega^t)$ is non-dominated and $\mathcal{Q}^T$ is also non-dominated.
- Indeed, if not, any dominating measure would have an uncountable number of atoms.

Conclusion

- We have understood in details the quasi-sure no arbitrage condition and studied the link with different types of robust no-arbitrage conditions (local or global) in discrete time.

Conclusion

- We have understood in details the quasi-sure no arbitrage condition and studied the link with different types of robust no-arbitrage conditions (local or global) in discrete time.
- Our main result gives the existence of a set of priors having the same polar sets than the original one and such each priors is arbitrage free.

Conclusion

- We have understood in details the quasi-sure no arbitrage condition and studied the link with different types of robust no-arbitrage conditions (local or global) in discrete time.
- Our main result gives the existence of a set of priors having the same polar sets than the original one and such each priors is arbitrage free.
- We give concrete examples where all the quantities appearing in the different definitions and characterizations of NA are explicit.

D. Bartl, P. Cheridito, and M. Kupper. Robust expected utility maximization with medial limits. *Journal of Mathematical Analysis and Applications*, 471(1-2) :752–775, 2019.

S. Biagini, B. Bouchard, C. Kardaras, and M. Nutz. Robust fundamental theorem for continuous processes. *Mathematical Finance*, 2015.

R. Blanchard and L. Carassus. Multiple-priors investment in discrete time for unbounded utility function. *Annals of Applied Probability*, 88(2) : 241–281, 2018.

B. Bouchard and M. Nutz. Arbitrage and duality in nondominated discrete-time models. *Annals of Applied Probability*, 25(2) :823–859, 2015.

M.H.A. Davis and D. Hobson. The range of traded option prices. *Mathematical Finance*, 17(1) :1–14, 2007.

J. Jacod and A. N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance Stochastic*, 2 :259–273, 1998.

J. Obłój and J. Wiesel. A unified framework for robust modelling of financial markets in discrete time. *arxiv*, 2018.

M. Rásonyi and A. Meireles-Rodrigues. On utility maximisation under model uncertainty in discrete-time markets. *arxiv*, 2018.

M. Rásonyi and L. Stettner. On the utility maximization problem in discrete-time financial market models. *Annals of Applied Probability*, 15 :1367–1395, 2005.