Optimal make-take fees for market making regulation

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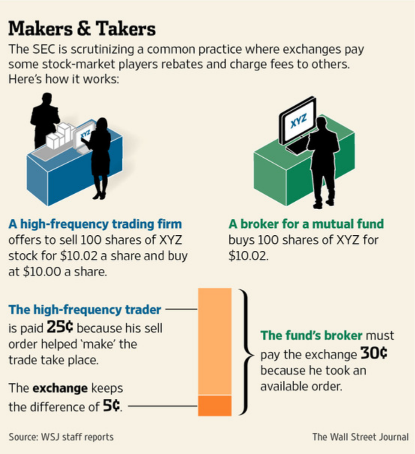
Trading Makers-Takers fees... towards Fintech

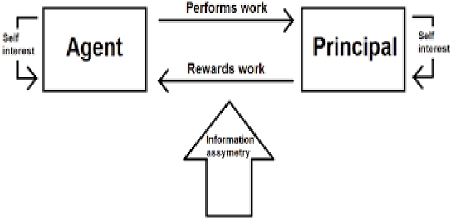


Delegation problem: accounting for moral hazard

X: value of an output process owned by Principal
Agent devotes effort a, thus impacting distribution of $X \Longrightarrow X^a$

- cost of effort $c(a)$
- compensation ξ : contract

Choose ξ so that Agent devotes effort in the interest of Principal



Second best contracting: Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A \quad := \max_a \mathbb{E}\, U_A\big(\qquad - c(a)\big)$$

(Static) Principal-Agent Problem

- Principal delegates management of output process X,
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  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

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- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

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Contract theory at the heart of modern economic theory

Jean Tirole, Nobel Prize 2014: organization theory, regulation

Oliver Hart and Bengt Holmström, Nobel Prize 2016

Holmström & Milgrom '85:

Principal-Agent problem more accessible in continuous time

Cvitanić & Zhang '12 (Book): calculus of variations...

Sannikov '08: continuation utility process, drift control

Cvitanić, Possamaï & NT '18: dynamic programming approach, finite horizon

Lin, Ren, Yang & NT '19: extension to random horizon

Principal-Agent problem: continuous time formulation

Agent problem:

$$\text{For } \xi \in \mathbb{L}^0(\Omega, \mathbb{R}), \quad V_0^A(\xi) \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi(X) - \int_0^T c_t(\nu_t)dt\Big]$$

$\mathbb{P} \in \mathcal{P}$: weak solution of Output process for some ν valued in U:

$$dX_t \quad = \quad b_t(X, \nu_t)dt + \sigma_t(X, \nu_t)dW_t^{\mathbb{P}} \quad \mathbb{P} - \text{a.s.}$$

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Principal problem

$$\text{Given solution } \mathbb{P}^*(\xi), \quad V_0^P := \sup_{\xi \in \Xi_\rho} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[U\big(X_T - \xi(X)\big)\Big]$$

where $\Xi_\rho := \{\xi(X_\cdot) : V_0^A(\xi) \geq \rho\}$

Extensions: random (possibly ∞) horizon, heterogeneous agents with possibly mean field interaction, competing Principals...

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Intuition from the Markov setting

If $\xi = g(X_T)$, then $V^A = v(0, X_0)$ where v solution of HJB equation

$$\partial_t v + H(Dv, D^2 v) = 0, \quad v\big|_{t=T} = g$$

- Hamiltonian $H(z,\gamma) := \sup_{u\in U}\left\{b(u)\cdot z + \frac{1}{2}\sigma_t\sigma_t^\top(u):\gamma - c_t(u)\right\}$
- optimal Agent response $u^* = \hat{u}(Dv, D^2 v)$ maximizer of H

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By Itô's formula, we may rewrite $g(X_T) = v(T, X_T)$ as

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$\Longrightarrow$ **Principal problem (of optimal choice of g) reduces to**

$$\max_{V^A \geq \rho} \max_{Z,\Gamma} \mathbb{E}\left[U\left(\ell(X_T) - g^{V^A, Z, \Gamma}(X_T)\right)\right]$$

where $(Z, \Gamma) = (v, Dv)$, s.t. v solves HJB $\Longrightarrow$ **difficult constraints...**

A subset of revealing contracts

Path-dependent Hamiltonian for the Agent problem

$$H_t(\omega, z, \gamma) \quad := \quad \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2}\sigma_t\sigma_t^\top(\omega, u) : \gamma - c_t(\omega, u) \right\}$$

For $Y_0 \in \mathbb{R}$, Z, Γ $\mathbb{F}^X -$ prog meas, define $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X \rangle_s - H_s(X, Z_s, \Gamma_s)ds$$

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Proposition

$V_0^A\big(Y_T^{Z,\Gamma}\big) = Y_0$. Moreover $\mathbb{P}^*$ is optimal iff

$$\nu_t^* = \underset{u \in U}{\mathrm{Argmax}}\, H_t(Z_t, \Gamma_t) = \hat{\nu}(Z_t, \Gamma_t)$$

Proof: classical verification argument !

For all $\mathbb{P} \in \mathcal{P}$, denote $J_A(\xi, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\big[\xi - \int_0^T c_t^\nu \, dt\big]$. Then

$$J_A\big(Y_T^{Z,\Gamma}, \mathbb{P}\big) = \mathbb{E}^{\mathbb{P}}\Big[Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(Z_t, \Gamma_t)dt - \int_0^T c_t^\nu \, dt\Big]$$

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with equality iff $\nu = \nu^*$ maximizes the Hamiltonian

Principal problem restricted to revealing contracts

$\Longrightarrow$ Principal's value function under revealing contracts

$$V_0^P \geq \sup_{Y_0 \geq \rho} V_0(X_0, Y_0), \quad V_0(X_0, Y_0) := \sup_{(Z,\Gamma)\in\mathcal{V}} \mathbb{E}\Big[U\big(X_T - Y_T^{Z,\Gamma}\big)\Big]$$

where $\mathcal{V} := \Big\{(Z,\Gamma): \; Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \; \mathcal{P}^*\big(Y_T^{Z,\Gamma}\big) \neq \emptyset\Big\}$

and the dynamics of the pair (X, Y) under "optimal response"

$$dX_t = b_t\big(X, \hat{\nu}(Z_t, \Gamma_t)\big)dt + \sigma_t\Big(X, \hat{\nu}(Z_t, \Gamma_t)\Big)dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Z_t, \Gamma_t)dt$$

(1 state augmented) controlled SDE with controls (Z, Γ)

Reduction to standard control problem

Theorem (Cvitanić, Possamaï & NT '15)

Assume $\mathcal{V} \neq \emptyset$. Then

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer Y_0^*, the corresponding optimal controls (Z^*, Γ^*) induce an optimal contract

$$\xi^* = Y_0^* + \int_0^T Z_t^* \cdot dX_t + \frac{1}{2}\Gamma_t^* : d\langle X\rangle_t - H_t(X, Z_t^*, \Gamma_t^*)dt$$

Recall the subclass of contracts

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

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To prove the main result, it suffices to prove the **representation**

$$\boxed{\text{for all } \xi \in ?? \quad \exists\,(Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma},\ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}}$$

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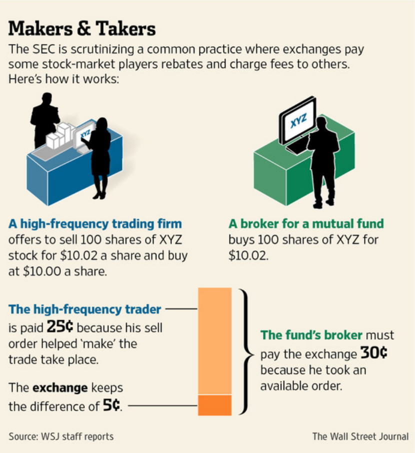
OR, weaker sufficient condition:

$$\boxed{\text{for all } \xi \in ?? \quad \exists\,(Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{“}Y_T^{Z^n,\Gamma^n} \longrightarrow \xi\text{”}}$$

Trading Makers-Takers fees... towards Fintech



Market makers, and brokers trading

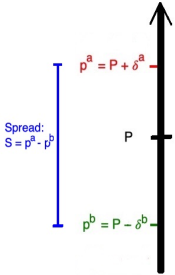
- Fundamental price $\{P_t\}_{t\geq 0}$: $dP_t = \sigma dW_t$

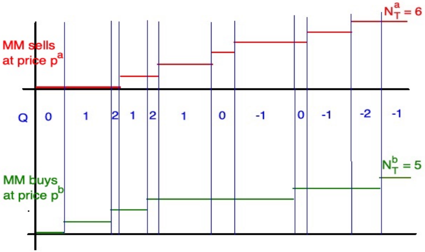
- **Market Maker** sets bid-ask prices $p^b_t = P_t - \delta^b_t$ and $p^a_t = P_t + \delta^a_t$

- N^b_t, N^a_t: # trades, unit jump point process with intensities

$$\lambda^b_t = \lambda(\delta^b_t) \quad \text{and} \quad \lambda^a_t = \lambda(\delta^a_t), \quad \text{with} \quad \lambda(x) = Ae^{-\frac{k}{\sigma}(x+c)}$$

$\Longrightarrow$ MM inventory $Q_t = N^b_t - N^a_t$, where





Market makers, and brokers trading

MM and Platform have constant absolute risk aversion

$$U_A(x) = -e^{-\gamma x}, \quad U_P(x) = -e^{-\eta x}$$

• MM chooses bid and ask prices:

$$V_A(\xi) \quad := \quad \sup_{\delta=(\delta^b,\delta^a)} \mathbb{E}^{\delta} U_A\Big(\xi + \int_0^T p_t^a dN_t^a - p_t^b dN_t^b + Q_T P_T\Big)$$

• Given optimal response $\delta^*(\xi)$, **Platform** chooses optimal contract

$$V_P \quad = \quad \sup_{\xi \in \Xi_R} \mathbb{E}^{\delta^*(\xi)} U_P\big(-\xi + c(N_T^a + N_T^b)\big)$$

c: fee paid by broker $\Longrightarrow$ c affects the arrival process...

Avellaneda & Stoikov '08 corresponds to $\xi = 0$

Optimal MM compensation

Let

$$u(t,q) = \sum_{p\ge 0} \frac{[C_1'(T-t)]^p}{p!} \sum_{j\ge 0} \frac{[C_1'(T-t)]^j}{j!} e^{-C_1(T-t)(q+j-p)^2} \mathbf{1}_{\{|q+j-p|\le \bar q\}},$$

with constants C_1, C_1'. Then,

Optimal contract is

$$\widehat{\xi} = U_A^{-1}(R) + \int_0^T \widehat{Z}_t^a dN_t^a + \widehat{Z}_t^b dN_t^b + \widehat{Z}_t^P dP_t + \Big(\frac{1}{2}\gamma\sigma^2\big(\hat{Z}_t^P + Q_t\big)^2 - H(\widehat{Z}_t, Q_t)\Big) dt$$

where $\widehat{Z}_t^P = \frac{-\gamma}{\eta+\gamma} Q_t$: inventory risk sharing

and $\widehat{Z}_t^i = c + \frac{1}{\eta}\Big[\ln\Big(\frac{u(t,Q_t)}{u(t,Q_t+\varepsilon_i)}\Big) - \zeta_0\Big]$, $i = b, a$, $\varepsilon_b = 1$, $\varepsilon_a = -1$,

$$\zeta_0 := -\log\Big(1 - \frac{1}{(1+\frac{k}{\sigma\gamma})(1+\frac{k}{\sigma\eta})}\Big)$$

Effect of the exchange optimal incentive policy

Parameters values from Guéant, Lehalle and Fernandez-Tapia:

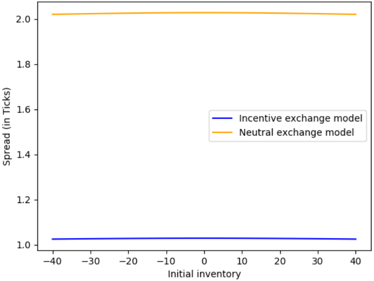
$$T = 600s, \quad \sigma = 0.3\text{Tick}.s^{-1/2}, \quad A = 0.9s^{-1}, \quad k = 0.3s^{-1/2},$$
$$\bar{q} = 50 \text{ unities}, \quad \gamma = 0.01\text{Tick}^{-1}, \quad \eta = 1\text{Tick}^{-1}, \quad c = 0.5\text{Tick}.$$

Impact of the incentive policy on the spread

The optimal spread is given by $\widehat{S}_t = \widehat{\delta}_t^a + \widehat{\delta}_t^b$ with

$$\widehat{\delta}_t^i = \delta_t^i(\widehat{\xi}) = -\widehat{Z}_t^i + \frac{1}{\gamma}\log\left(1 + \frac{\sigma\gamma}{k}\right), \quad i = a, b$$

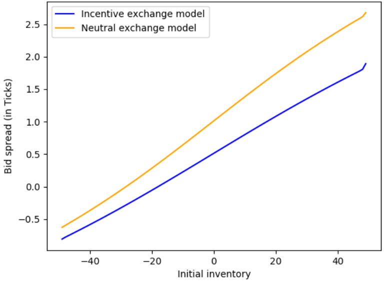
Incentive contract induces spread to be cut by half

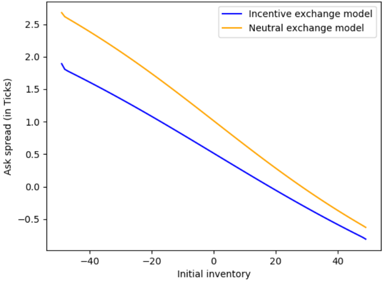


Optimal initial spread with/without the exchange incentive policy in terms of initial inventory Q_0.

Impact of the incentive policy on the spread

Incentive contract induces bid and ask spreads to be cut by half

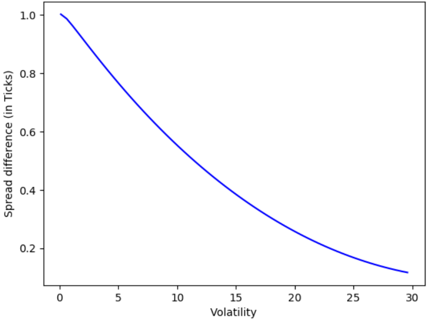




Optimal initial bid (left) and ask (right) spread component with/without the exchange incentive policy in terms of initial inventory Q_0.

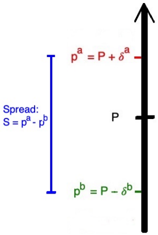
Impact of the volatility on the incentive policy

Incentive contract effect decreases with volatility...



Initial optimal spread difference (with/without incentive) in terms of the volatility σ.

Regulation implication: how to choose the constant fee c



Bid-ask spread $\widehat{S}_t$ is explicit...

Assume Exchange fixes the transaction cost c so that $\widehat{S}_t = 1$

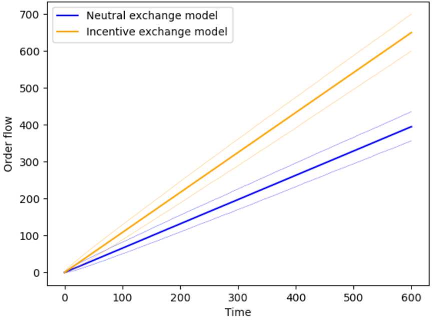
Then, we compute that

$$c \approx \frac{\sigma}{k} - \frac{1}{2}\text{Tick}$$

Impact of the incentive policy on the market liquidity

transactions $= N_T^a + N_T^b$

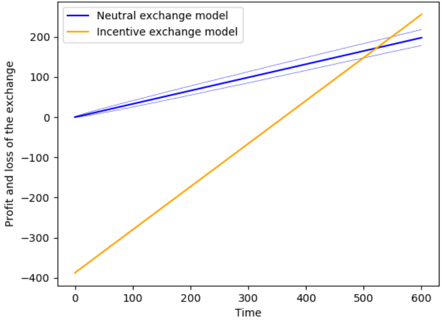
Incentive contract induces more transactions...



Average order flow with 95% confidence interval with/without incentive policy (5000 scenarios).

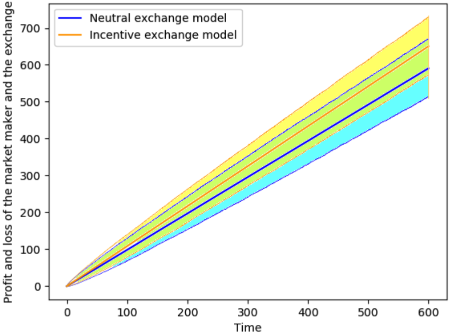
Impact of the incentive policy on the platform P&L

$$-\hat{\xi} + c(N_T^a + N_T^b)$$



platform P&L with 95% confidence interval
with/without incentive policy (5000 scenarios).

Impact of the incentive policy on the market maker and exchange profit and loss



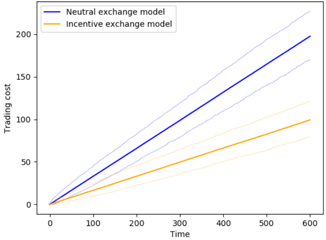
Aggregate P&L of MM and exchange with 95% confidence interval with/without incentive policy (5000 scenarios).

Impact of the incentive policy on trading costs

- One market taker buying a fixed quantity $Q_{final} = 200$ shares

trading cost $\int_0^T \delta_s^a dN_s^a$. with or without incentive

Incentive contract decreases significantly the average trading cost



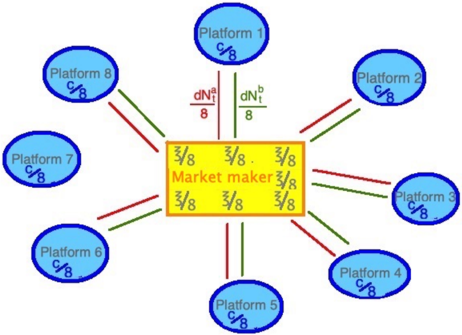
Average trading cost with 95% confidence interval with/without incentive policy (5000 scenarios).

Summarizing the benefits from optimal contracting

Benefits of the exchange incentive policy

- Smaller spread.
- Increase of the market liquidity.
- Increase of the profit and loss of the MM and the exchange.
- Less transaction costs.

Symmetric platforms in Nash equilibrium



1 Market maker
facing n symmetric platforms

Symmetric platforms in Nash equilibrium

• MM chooses bid and ask prices:

$$V_A(\xi) \quad := \quad \sup_{\delta=(\delta^b,\delta^a)} \mathbb{E}^{\delta} U_A\Big(\xi + \int_0^T p_t^a dN_t^a - p_t^b dN_t^b + Q_T P_T\Big)$$

where $\xi = \xi_1 + \ldots + \xi_n$

• Given optimal response $\delta^*(\xi)$, **Platform** i chooses optimal contract ξ_i, given $\tilde{\xi} := \sum_{j \neq i} \xi_j$:

$$V_P \quad = \quad \sup_{\xi_i \in \Xi_R(\tilde{\xi})} \mathbb{E}^{\delta^*(\xi_i+\tilde{\xi})} U_P\Big(-\xi_i + \frac{c}{n}(N_T^a + N_T^b)\Big)$$

$\Longrightarrow$ Optimal contract $\xi_0^*(\tilde{\xi})$... independent of i

Symmetric platforms in Nash equilibrium

Nash Equilibrium

$(\xi_1, \ldots, \xi_n)$ is a Nash equilibrium if

$$\xi_0^*\big(\textstyle\sum_{j \neq i} \xi_j\big) = \xi_i, \quad \text{for all} \quad i = 1, \ldots, n$$

A Nash equilibrium $(\xi_1, \ldots, \xi_n)$ is symmetric if $\xi_1 = \cdots = \xi_n$

For a symmetric Nash equilibrium, we must solve

$$\xi_0^*\big((n-1)\xi_0\big) = \xi_0$$

If $\hat{\xi}_0$ defines a symmetric Nash equilibrium, then the Market maker receives the total payment $\hat{\xi}^{(n)} := n\hat{\xi}_0$.

Optimal MM compensation

Let

$$u_n(t,q) = \sum_{p\geq 0} \frac{[C_n'(T-t)]^p}{p!} \sum_{j\geq 0} \frac{[C_n'(T-t)]^j}{j!} e^{-C_n(T-t)(q+j-p)^2} \mathbf{1}_{\{|q+j-p|\leq \bar{q}\}}$$

with constants

$$C_n = \frac{k\sigma}{2(\frac{n}{\eta}+\frac{1}{\gamma})}, \quad C_n' = C_1'(\alpha,\beta)e^{(n-1)\beta} \frac{(1+n\frac{\beta}{\alpha})+\beta}{(1+\frac{\beta}{\alpha})+\beta}$$

and $\alpha := \frac{k}{\sigma\gamma}, \quad \beta := \frac{k}{\sigma\eta}, \quad C_1'(\alpha,\beta) := A\beta\left(1+\frac{1}{\alpha}\right)^{-\alpha}\left(1-\frac{1}{(1+\alpha)(1+\beta)}\right)^{1+\beta}$

Optimal MM compensation: main result

Theorem

There exists a unique symmetric Nash equilibrium with optimal contract

$$\widehat{\xi}^{(n)} \;=\; U_A^{-1}(R) + \int_0^T \widehat{Z}_t^{n,a} dN_t^a + \widehat{Z}_t^{n,b} dN_t^b + \widehat{Z}_t^{n,P} dP_t + \Big(\frac{1}{2}\gamma\sigma^2\big(\hat{Z}_t^{n,P} + Q_t\big)^2 - H\big(\widehat{Z}_t^n, Q_t\big)\Big)dt$$

$\widehat{Z}_t^{n,P} = \frac{-n\gamma}{\eta+n\gamma} Q_t$: inventory risk sharing $\underset{n\to\infty}{\longrightarrow} -Q_t$ (Selling firm effect)

and $\widehat{Z}_t^{n,i} = c + \frac{n}{\eta}\Big[\ln\Big(\frac{u_n(t,Q_t)}{u_n(t,Q_t+\varepsilon_i)}\Big) - \zeta_0\Big], \quad i = b, a \quad \varepsilon_b = 1, \; \varepsilon_a = -1,$