Optimal make-take fees for market making regulation

Nizar Touzi

Ecole Polytechnique, France

with O. El Euch, T. Mastrolia, M. Rosenbaum

Paris, January 31, 2020
Trading Makers-Takers fees... towards Fintech

Makers & Takers
The SEC is scrutinizing a common practice where exchanges pay some stock-market players rebates and charge fees to others. Here’s how it works:

A high-frequency trading firm offers to sell 100 shares of XYZ stock for $10.02 a share and buy at $10.00 a share.

The high-frequency trader is paid 25¢ because his sell order helped ‘make’ the trade take place.

The exchange keeps the difference of 5¢.

A broker for a mutual fund buys 100 shares of XYZ for $10.02.

The fund’s broker must pay the exchange 30¢ because he took an available order.

Source: WSJ staff reports
The Wall Street Journal
Delegation problem: accounting for moral hazard

\( X \): value of an output process owned by Principal
Agent devotes effort \( a \), thus impacting distribution of \( X \mapsto X^a \)
- cost of effort \( c(a) \)
- compensation \( \xi : \) contract

Choose \( \xi \) so that Agent devotes effort in the interest of Principal
Second best contracting: Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by

$$V_A := \max_a E U_A(X^a - c(a))$$
(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by
  
  $$V_A(\xi) := \max_a \mathbb{E} U_A(\xi(X^a) - c(a)) \implies \hat{a}(\xi)$$
(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by
  \[
  V_A(\xi) := \max_a \mathbb{E} U_A(\xi(X^a) - c(a)) \implies \hat{a}(\xi)
  \]

- Principal chooses optimal contract by solving
  \[
  \max_{\xi} \mathbb{E} U_P(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})) \quad \text{under constraint} \quad V_A(\xi) \geq R
  \]
(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by
  
  $$V_A(\xi) := \max_a \mathbb{E} U_A(\xi(X^a) - c(a)) \implies \hat{a}(\xi)$$

- Principal chooses optimal contract by solving
  
  $$\max_\xi \mathbb{E} U_P(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})) \quad \text{under constraint} \quad V_A(\xi) \geq R$$
Contract theory at the heart of modern economic theory

Jean Tirole, Nobel Prize 2014: organization theory, regulation

Oliver Hart and Bengt Holmström, Nobel Prize 2016

Holmström & Milgrom '85:

Principal-Agent problem more accessible in continuous time

Cvitanić & Zhang '12 (Book): calculus of variations...

Sannikov '08: continuation utility process, drift control

Cvitanić, Possamaï & NT '18: dynamic programming approach, finite horizon

Lin, Ren, Yang & NT '19: extension to random horizon
Agent problem:

For $\xi \in L^0(\Omega, \mathbb{R})$, $V_A^0(\xi) := \sup_{P \in \mathcal{P}} \mathbb{E}^P \left[ \xi(X) - \int_0^T c_t(\nu_t)dt \right]$

$\mathbb{P} \in \mathcal{P}$: weak solution of Output process for some $\nu$ valued in $U$:

$$dX_t = b_t(X, \nu_t)dt + \sigma_t(X, \nu_t)dW^P_t \quad \mathbb{P} - \text{a.s.}$$
Agent problem:

For $\xi \in L^0(\Omega, \mathbb{R})$, $V_A^0(\xi) := \sup_{P \in \mathcal{P}} \mathbb{E}^P \left[ \xi(X) - \int_0^T c_t(\nu_t) dt \right]$

$\mathcal{P} \in \mathcal{P}$: weak solution of Output process for some $\nu$ valued in $U$:

$$dX_t = b_t(X, \nu_t) dt + \sigma_t(X, \nu_t) dW_t^P \quad P - \text{a.s.}$$

Principal problem

Given solution $P^*(\xi)$, $V_P^0 := \sup_{\xi \in \Xi_\rho} \mathbb{E}^{P^*}(\xi) \left[ U(X_T - \xi(X)) \right]$

where $\Xi_\rho := \{\xi(X) : V_A^0(\xi) \geq \rho\}$

Extensions: random (possibly $\infty$) horizon, heterogeneous agents with possibly mean field interaction, competing Principals...
Principal-Agent problem: continuous time formulation

Agent problem:

For $\xi \in L^0(\Omega, \mathbb{R})$, $V_0^A(\xi) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P} \left[ \xi(X) - \int_0^T c_t(\nu_t) dt \right]$

$\mathbb{P} \in \mathcal{P}$: weak solution of Output process for some $\nu$ valued in $U$:

$$dX_t = \sigma_t(X, \beta_t) [\lambda_t(X, \alpha_t) dt + dW^\mathbb{P}_t] \quad \mathbb{P} \text{ a.s.}$$

Principal problem

Given solution $\mathbb{P}^*(\xi)$, $V_0^P := \sup_{\xi \in \Xi_\rho} \mathbb{E}^{\mathbb{P}^*} \left[ U(X_T - \xi(X)) \right]$

where $\Xi_\rho := \{\xi(X.) : V_0^A(\xi) \geq \rho\}$

Extensions: random (possibly $\infty$) horizon, heterogeneous agents with possibly mean field interaction, competing Principals...
Intuition from the Markov setting

If $\xi = g(X_T)$, then $V^A = \nu(0, X_0)$ where $\nu$ solution of HJB equation

$$\partial_t \nu + H(D\nu, D^2\nu) = 0, \quad \nu\big|_{t=\tau} = g$$

- Hamiltonian $H(z, \gamma) := \sup_{u \in U} \{ b(u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top(u) : \gamma - c_t(u) \}$
- optimal Agent response $u^* = \hat{u}(D\nu, D^2\nu)$ maximizer of $H$
Intuition from the Markov setting

If $\xi = g(X_T)$, then $V^A = \nu(0, X_0)$ where $\nu$ solution of HJB equation

$$\partial_t \nu + H(D\nu, D^2\nu) = 0, \quad \nu|_{t=T} = g$$

- Hamiltonian $H(z, \gamma) := \sup_{u \in U} \{ b(u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top (u) : \gamma - c_t(u) \}$
- optimal Agent response $u^* = \hat{u}(D\nu, D^2\nu)$ maximizer of $H$

By Itô’s formula, we may rewrite $g(X_T) = \nu(T, X_T)$ as

$$g(X_T) = V^A + \int_0^T D\nu(t, X_t) dX_t + \frac{1}{2} D^2\nu(t, X_t) : d\langle X \rangle_t + \partial_t \nu(t, X_t) dt$$
Intuition from the Markov setting

If \( \xi = g(X_T) \), then \( V^A = v(0, X_0) \) where \( v \) solution of HJB equation

\[
\partial_t v + H(Dv, D^2 v) = 0, \quad v|_{t=T} = g
\]

- Hamiltonian \( H(z, \gamma) := \sup_{u \in U} \{ b(u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top(u) : \gamma - c_t(u) \} \)
- optimal Agent response \( u^* = \hat{u}(Dv, D^2 v) \) maximizer of \( H \)

By Itô’s formula, we may rewrite \( g(X_T) = v(T, X_T) \) as

\[
g(X_T) = V^A + \int_0^T Dv(t, X_t) dX_t + \frac{1}{2} D^2 v(t, X_t) : d\langle X \rangle_t - H(Dv, D^2 v)(t, X_t) dt
\]

\[
= V^A + \int_0^T Z(t, X_t) dX_t + \frac{1}{2} \Gamma(t, X_t) : d\langle X \rangle_t - H(Z, \Gamma)(t, X_t) dt
\]
Intuition from the Markov setting

If $\xi = g(X_T)$, then $V^A = \nu(0, X_0)$ where $\nu$ solution of HJB equation

$$\partial_t \nu + H(D\nu, D^2 \nu) = 0, \quad \nu|_{t=T} = g$$

- Hamiltonian $H(z, \gamma) := \sup_{u \in U} \{ b(u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top (u) : \gamma - c_t(u) \}$
- optimal Agent response $u^* = \hat{u}(D\nu, D^2 \nu)$ maximizer of $H$

By Itô’s formula, we may rewrite $g(X_T) = \nu(T, X_T)$ as

$$g(X_T) = V^A + \int_0^T D\nu(t, X_t) dX_t + \frac{1}{2} D^2 \nu(t, X_t) : d\langle X \rangle_t - H(D\nu, D^2 \nu)(t, X_t) dt$$

$$= V^A + \int_0^T Z(t, X_t) dX_t + \frac{1}{2} \Gamma(t, X_t) : d\langle X \rangle_t - H(Z, \Gamma)(t, X_t) dt$$

$\implies$ Principal problem (of optimal choice of $g$) reduces to

$$\max_{V^A \geq \rho} \max_{Z, \Gamma} \mathbb{E} \left[ U(\ell(X_T) - g^{V^A, Z, \Gamma}(X_T)) \right]$$

where $(Z, \Gamma) = (\nu, D\nu)$, s.t. $\nu$ solves HJB $\implies$ difficult constraints...
A subset of revealing contracts

Path-dependent Hamiltonian for the Agent problem

\[ H_t(\omega, z, \gamma) := \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t^T \sigma_t (\omega, u) : \gamma - c_t(\omega, u) \right\} \]

For \( Y_0 \in \mathbb{R}, Z, \Gamma \in \mathcal{F}^X \) — prog meas, define \( \mathbb{P}-a.s. \) for all \( \mathbb{P} \in \mathcal{P} \)

\[ Y_t^{Z, \Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Z_s, \Gamma_s)ds \]
A subset of revealing contracts

Path-dependent Hamiltonian for the Agent problem

\[
H_t(\omega, z, \gamma) := \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top (\omega, u) : \gamma - c_t(\omega, u) \right\}
\]

For \( Y_0 \in \mathbb{R}, Z, \Gamma \mathbb{F}^X \) – prog meas, define \( \mathbb{P} \)-a.s. for all \( \mathbb{P} \in \mathcal{P} \)

\[
Y_t^{Z, \Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Z_s, \Gamma_s)ds
\]

Proposition

\( V_0^A(Y_T^{Z, \Gamma}) = Y_0 \). Moreover \( \mathbb{P}^* \) is optimal iff

\[
\nu_t^* = \arg \max_{u \in U} H_t(Z_t, \Gamma_t) = \hat{\nu}(Z_t, \Gamma_t)
\]
Proof: classical verification argument!

For all \( \mathbb{P} \in \mathcal{P} \), denote \( J_A(\xi, \mathbb{P}) := \mathbb{E}^\mathbb{P}\left[\xi - \int_0^T c_t \, dt\right] \). Then

\[
J_A(Y_T^{Z,\Gamma}, \mathbb{P}) = \mathbb{E}^\mathbb{P}\left[Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t(Z_t, \Gamma_t) \, dt - \int_0^T c_t' \, dt\right]
\]
Proof: classical verification argument!

For all $P \in \mathcal{P}$, denote $J_A(\xi, P) := \mathbb{E}^P[\xi - \int_0^T c_t' dt]$. Then

$$J_A(Y_T^{Z, \Gamma}, P) = \mathbb{E}^P \left[ Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t(Z_t, \Gamma_t) dt - \int_0^T c_t' dt \right]$$

$$= Y_0 + \mathbb{E}^P \int_0^T \left\{ b_t' \cdot Z_t + \frac{1}{2} \sigma \sigma^\top : \Gamma_t - c_t' - H_t(Z_t, \Gamma_t) \right\} dt$$
Proof: classical verification argument!

For all $\mathbb{P} \in \mathcal{P}$, denote $J_A(\xi, \mathbb{P}) := \mathbb{E}^\mathbb{P} [\xi - \int_0^T c_t' \, dt]$. Then

$$J_A(Y_T^Z, \Gamma, \mathbb{P}) = \mathbb{E}^\mathbb{P} \left[ Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : \langle X \rangle_t - H_t(Z_t, \Gamma_t) \, dt - \int_0^T c_t' \, dt \right]$$

$$= Y_0 + \mathbb{E}^\mathbb{P} \int_0^T \left\{ b_t' \cdot Z_t + \frac{1}{2} \sigma \sigma^\top : \Gamma_t - c_t' - H_t(Z_t, \Gamma_t) \right\} \, dt$$

$$\leq Y_0 \quad \text{by definition of } H$$
Proof: classical verification argument!

For all $\mathbb{P} \in \mathcal{P}$, denote $J_A(\xi, \mathbb{P}) := \mathbb{E}^\mathbb{P}[\xi - \int_0^T c_t \nu dt]$. Then

$$J_A(Y_T^Z, \Gamma, \mathbb{P}) = \mathbb{E}^\mathbb{P}\left[Y_0 + \int_0^T Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t(Z_t, \Gamma_t) dt - \int_0^T c_t \nu dt\right]$$

$$= Y_0 + \mathbb{E}^\mathbb{P}\int_0^T \left\{ b_t^\nu \cdot Z_t + \frac{1}{2} \sigma \sigma^\top : \Gamma_t - c_t^\nu - H_t(Z_t, \Gamma_t) \right\} dt$$

$$\leq Y_0 \quad \text{by definition of } H$$

with equality iff $\nu = \nu^*$ maximizes the Hamiltonian.
Principal problem restricted to revealing contracts

$V^P_0 \geq \sup_{Y_0 \geq \rho} V_0(X_0, Y_0), \quad V_0(X_0, Y_0) := \sup_{(Z, \Gamma) \in \mathcal{V}} \mathbb{E}\left[U(X_T - Y^Z_T, \Gamma)\right]$ 

where $\mathcal{V} := \{(Z, \Gamma) : Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \mathcal{P}^*(Y^Z_T, \Gamma) \neq \emptyset\}$

and the dynamics of the pair $(X, Y)$ under “optimal response”

$$dX_t = b_t(X, \hat{\nu}(Z_t, \Gamma_t)) \, dt + \sigma_t(X, \hat{\nu}(Z_t, \Gamma_t)) \, dW_t$$

$$dY^Z_t = Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t(X, Z_t, \Gamma_t) \, dt$$

(1 state augmented) controlled SDE with controls $(Z, \Gamma)$
Reduction to standard control problem

Theorem (Čvitanić, Possamaï & NT ’15)

Assume $\mathcal{V} \neq \emptyset$. Then

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer $Y_0^*$, the corresponding optimal controls $(Z^*, \Gamma^*)$ induce an optimal contract

$$\xi^* = Y_0^* + \int_0^T Z_t^* \cdot dX_t + \frac{1}{2} \Gamma_t^* : d\langle X \rangle_t - H_t(X, Z_t^*, \Gamma_t^*) dt$$
Recall the subclass of contracts

\[ Y_{t}^{Z,\Gamma} = Y_{0} + \int_{0}^{t} Z_{s} \cdot dX_{s} + \frac{1}{2} \Gamma_{s} : d\langle X \rangle_{s} - H_{s}(X, Y_{s}^{Z,\Gamma}, Z_{s}, \Gamma_{s})ds \]

\[ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \]
Recall the subclass of contracts

\[ Y_{t}^{Z,\Gamma} = Y_{0} + \int_{0}^{t} Z_{s} \cdot dX_{s} + \frac{1}{2} \Gamma_{s} : d\langle X \rangle_{s} - H_{s}(X, Y_{s}^{Z,\Gamma}, Z_{s}, \Gamma_{s})ds \]

\[ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \]

To prove the main result, it suffices to prove the representation

for all \( \xi \in \mathbb{R} \) \( \exists (Y_{0}, Z, \Gamma) \text{ s.t. } \xi = Y_{T}^{Z,\Gamma}, \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \)
Recall the subclass of contracts

\[ Y_t^{Z, \Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Y_s^{Z, \Gamma}, Z_s, \Gamma_s) \, ds \]

\( \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \)

To prove the main result, it suffices to prove the representation

for all \( \xi \in \mathbb{R} \) \( \exists (Y_0, Z, \Gamma) \) s.t. \( \xi = Y_T^{Z, \Gamma} \), \( \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \)

OR, weaker sufficient condition:

for all \( \xi \in \mathbb{R} \) \( \exists (Y_0^n, Z^n, \Gamma^n) \) s.t. “\( Y_T^{Z^n, \Gamma^n} \rightarrow \xi \)”
Trading Makers-Takers fees... towards Fintech

**Makers & Takers**

The SEC is scrutinizing a common practice where exchanges pay some stock-market players rebates and charge fees to others. Here’s how it works:

- **A high-frequency trading firm** offers to sell 100 shares of XYZ stock for $10.02 a share and buy at $10.00 a share.

- **A broker for a mutual fund** buys 100 shares of XYZ for $10.02.

**The high-frequency trader** is paid **25¢** because his sell order helped ‘make’ the trade take place.

**The exchange** keeps the difference of **5¢**.

**The fund’s broker** must pay the exchange **30¢** because he took an available order.

Source: WSJ staff reports

The Wall Street Journal
Market makers, and brokers trading

- Fundamental price $\{P_t\}_{t \geq 0}$: $dP_t = \sigma dW_t$

- Market Maker sets bid-ask prices $p^b_t = P_t - \delta^b_t$ and $p^a_t = P_t + \delta^a_t$

- $N^b_t, N^a_t$: # trades, unit jump point process with intensities

$$\lambda^b_t = \lambda(\delta^b_t) \quad \text{and} \quad \lambda^a_t = \lambda(\delta^a_t), \quad \text{with} \quad \lambda(x) = Ae^{-\frac{k}{\sigma}(x+c)}$$

$\implies$ MM inventory $Q_t = N^b_t - N^a_t$, where
Market makers, and brokers trading

MM and Platform have constant absolute risk aversion

\[ U_A(x) = -e^{-\gamma x}, \quad U_P(x) = -e^{-\eta x} \]

- MM chooses bid and ask prices:

\[ V_A(\xi) := \sup_{\delta = (\delta^b, \delta^a)} \mathbb{E}^\delta U_A(\xi + \int_0^T p^a_t dN^a_t - p^b_t dN^b_t + Q_T P_T) \]

- Given optimal response \( \delta^*(\xi) \), Platform chooses optimal contract

\[ V_P = \sup_{\xi \in \Xi_R} \mathbb{E}^{\delta^*(\xi)} U_P(-\xi + c (N^a_T + N^b_T)) \]

\( c \) : fee paid by broker \( \implies \) \( c \) affects the arrival process...

Avellaneda & Stoikov '08 corresponds to \( \xi = 0 \)
Optimal MM compensation

Let
\[ u(t, q) = \sum_{p \geq 0} \frac{[C'_1(T-t)]^p}{p!} \sum_{j \geq 0} \frac{[C'_1(T-t)]^j}{j!} e^{-C_1(T-t)(q+j-p)^2} 1_{|q+j-p| \leq \bar{q}}, \]

with constants \( C_1, C'_1 \). Then,

**Optimal contract is**

\[
\hat{\xi} = U_A^{-1}(R) + \int_0^T \hat{Z}_t^a dN_t^a + \hat{Z}_t^b dN_t^b + \hat{Z}_t^P dP_t \\
+ \left( \frac{1}{2} \gamma \sigma^2 (\hat{Z}_t^P + Q_t)^2 - H(\hat{Z}_t, Q_t) \right) dt
\]

where \( \hat{Z}_t^P = \frac{-\gamma}{\eta + \gamma} Q_t \): inventory risk sharing

and \( \hat{Z}_t^i = c + \frac{1}{\eta} \left[ \ln \left( \frac{u(t, Q_t)}{u(t, Q_t + \varepsilon_i)} \right) - \zeta_0 \right], \ i = b, a, \ \varepsilon_b = 1, \ \varepsilon_a = -1, \)

\[ \zeta_0 := -\log \left( 1 - \frac{1}{(1 + \frac{k}{\sigma \gamma})(1 + \frac{k}{\sigma \eta})} \right) \]
Effect of the exchange optimal incentive policy

Parameters values from Guéant, Lehalle and Fernandez-Tapia:

\[ T = 600s, \quad \sigma = 0.3\text{Tick.s}^{-1/2}, \quad A = 0.9s^{-1}, \quad k = 0.3s^{-1/2}, \]
\[ \bar{q} = 50 \text{ unities}, \quad \gamma = 0.01\text{Tick}^{-1}, \quad \eta = 1\text{Tick}^{-1}, \quad c = 0.5\text{Tick}. \]
Impact of the incentive policy on the spread

The optimal spread is given by \( \hat{S}_t = \hat{\delta}_t^a + \hat{\delta}_t^b \) with

\[
\hat{\delta}_t^i = \delta_t^i(\hat{\xi}) = -\hat{Z}_t^i + \frac{1}{\gamma} \log \left( 1 + \frac{\sigma\gamma}{k} \right), \quad i = a, b
\]

Incentive contract induces spread to be cut by half

Optimal initial spread with/without the exchange incentive policy in terms of initial inventory \( Q_0 \).
Impact of the incentive policy on the spread

Incentive contract induces bid and ask spreads to be cut by half

Optimal initial bid (left) and ask (right) spread component with/without the exchange incentive policy in terms of initial inventory $Q_0$. 
Impact of the volatility on the incentive policy

Incentive contract effect decreases with volatility...

Initial optimal spread difference (with/without incentive) in terms of the volatility $\sigma$. 
Regulation implication: how to choose the constant fee $c$

Bid-ask spread $\hat{S}_t$ is explicit...

**Assume** Exchange fixes the transaction cost $c$ so that $\hat{S}_t = 1$

Then, we compute that

$$c \approx \frac{\sigma}{k} - \frac{1}{2} \text{Tick}$$
Impact of the incentive policy on the market liquidity

\[ \# \text{ transactions} = N^a_T + N^b_T \]

Incentive contract induces more transactions...

Average order flow with 95\% confidence interval with/without incentive policy (5000 scenarios).
Impact of the incentive policy on the platform P&L

$$-\hat{\xi} + c(N^a_T + N^b_T)$$

platform P&L with 95% confidence interval with/without incentive policy (5000 scenarios).
Impact of the incentive policy on the market maker and exchange profit and loss

Aggregate P&L of MM and exchange with 95% confidence interval with/without incentive policy (5000 scenarios).
Impact of the incentive policy on trading costs

- One market taker buying a fixed quantity $Q_{\text{final}} = 200$ shares

**trading cost** $\int_0^T \delta_s^a dN_s^a$. with or without incentive

Incentive contract decreases significantly the average trading cost

![Graph showing average trading cost with 95% confidence interval with/without incentive policy (5000 scenarios).](image)

Average trading cost with 95% confidence interval with/without incentive policy (5000 scenarios).
Summarizing the benefits from optimal contracting

Benefits of the exchange incentive policy

- Smaller spread.
- Increase of the market liquidity.
- Increase of the profit and loss of the MM and the exchange.
- Less transaction costs.
Symmetric platforms in Nash equilibrium

1 Market maker facing $n$ symmetric platforms
Symmetric platforms in Nash equilibrium

- MM chooses bid and ask prices:

\[
V_A(\xi) := \sup_{\delta=(\delta^b, \delta^a)} E^\delta U_A\left(\xi + \int_0^T p^a_t dN^a_t - p^b_t dN^b_t + Q_T P_T\right)
\]

where \(\xi = \xi_1 + \ldots + \xi_n\)

- Given optimal response \(\delta^*(\xi)\), Platform \(i\) chooses optimal contract \(\xi_i\), given

\[\bar{\xi} := \sum_{j \neq i} \xi_j\]

\[
V_P = \sup_{\xi_i \in \Xi_R(\bar{\xi})} E^{\delta^*(\xi_i+\bar{\xi})} U_P\left(-\xi_i + \frac{c}{n} (N^a_T + N^b_T)\right)
\]

\[\implies\] Optimal contract \(\xi_0^*(\bar{\xi})\) ... independent of \(i\)
Symmetric platforms in Nash equilibrium

Nash Equilibrium

\((\xi_1, \ldots, \xi_n)\) is a Nash equilibrium if

\[\xi_i^* \left( \sum_{j \neq i} \xi_j \right) = \xi_i, \quad \text{for all } i = 1, \ldots, n\]

A Nash equilibrium \((\xi_1, \ldots, \xi_n)\) is symmetric if \(\xi_1 = \cdots = \xi_n\)

For a symmetric Nash equilibrium, we must solve

\[\xi_0^* ((n - 1)\xi_0) = \xi_0\]

If \(\hat{\xi}_0\) defines a symmetric Nash equilibrium, then the Market maker receives the total payment \(\hat{\xi}^{(n)} := n\hat{\xi}_0\).
Optimal MM compensation

Let

\[ u_n(t, q) = \sum_{p \geq 0} \frac{[C_n'(T-t)]^p}{p!} \sum_{j \geq 0} \frac{[C_n'(T-t)]^j}{j!} e^{-C_n(T-t)(q+j-p)^2} 1_{\{|q+j-p| \leq q\}} \]

with constants

\[ C_n = \frac{k \sigma}{2\left(\frac{n}{\eta} + \frac{1}{\gamma}\right)}, \quad C_n' = C_1'(\alpha, \beta)e^{(n-1)\beta} \frac{(1 + n\frac{\beta}{\alpha}) + \beta}{(1 + \frac{\beta}{\alpha}) + \beta} \]

and \( \alpha := \frac{k}{\sigma\gamma}, \quad \beta := \frac{k}{\sigma\eta}, \quad C_1'(\alpha, \beta) := A\beta \left(1 + \frac{1}{\alpha}\right)^{-\alpha} \left(1 - \frac{1}{(1 + \alpha)(1 + \beta)}\right)^{1+\beta} \)
Optimal MM compensation: main result

**Theorem**

There exists a unique symmetric Nash equilibrium with optimal contract

\[
\hat{\xi}^{(n)} = U_A^{-1}(R) + \int_0^T \hat{Z}^{n,a}_t dN^a_t + \hat{Z}^{n,b}_t dN^b_t + \hat{Z}^{n,P}_t dP_t
\]

\[+ \left( \frac{1}{2} \gamma \sigma^2 \left( \hat{Z}^{n,P}_t + Q_t \right)^2 - H(\hat{Z}_t^n, Q_t) \right) dt\]

\[
\hat{Z}^{n,P}_t = \frac{-n\gamma}{\eta + n\gamma} Q_t: \text{inventory risk sharing}\quad \overset{n \to \infty}{\longrightarrow} -Q_t \quad \text{(Selling firm effect)}
\]

and

\[
\hat{Z}^{n,i}_t = c + \frac{n}{\eta} \left[ \ln \left( \frac{u_n(t, Q_t)}{u_n(t, Q_t + \varepsilon_i)} \right) - \zeta_0 \right], \quad i = b, a \quad \varepsilon_b = 1, \quad \varepsilon_a = -1,
\]