Risk bounds under dependence uncertainty

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Outline

1. VaR bounds with marginal information
2. Reduced bounds,higher dimensional marginals
3. Variance and higher order moment constraints
4. Improved standard bounds
   - 4.1 One-sided dependence information
   - 4.2 (Partial) independence structures
   - 4.3 Two-sided improved bounds
5. Partially specified risk factor models
6. Subgroup structure models
7. Conclusion

1. VaR bounds with marginal information

Stochastic Dependence

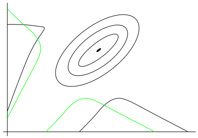
a) dependence modelling

$X = (X_1, \dots, X_d), \quad X_i \in \mathbb{R}^d$
$X_i \sim P_i \quad$ marginal structure

dependence structure: **Copula**

$\rightarrow$ copula models
Sklar's theorem

$f_\vartheta \sim P_\vartheta$



—: f_ϑ marginals;
—: given marginals

b) Hoeffding–Fréchet bounds

stochastic ordering, extremal dependence bounds for risk functionals

Conferences: *Probability with given marginals*
Rome 1990, Seattle 1993, Prague 1996, Barcelona 1998, Montreal 2004, Tartu 2007, Sao Paulo 2010

$X = (X_1, \ldots, X_n)$ risk vector

marginal information: $X_i \sim F_i$

$\longrightarrow$ high model risk for VaR, TVaR, ...

maximal upper risk

$$\boxed{M(s) = \sup_{X_i \sim F_i} \left\{ P\left(\sum_{i=1}^{n} X_i \geq s \right) \right\}}$$

$\underline{\mathrm{VaR}}_\alpha(S_n)$	$\mathrm{VaR}_\alpha(S_n^\perp)$	$\mathrm{VaR}_\alpha(S_n^c)$	$\overline{\mathrm{VaR}}_\alpha(S_n)$
lower bound	independent	comonotonic	upper bound

Theorem (unconstrained bounds)

$$A := \sum_{i=1}^{n} \mathrm{LTVaR}_\alpha(X_i) = \mathrm{LTVaR}_\alpha(S_n^c)$$

$$\leq \mathrm{VaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n)$$

$$\leq \mathrm{TVaR}_\alpha(S_n^c) = \sum_{i=1}^{n} \mathrm{TVaR}_\alpha(X_i) =: B$$

$$\mathrm{LTVaR}_\alpha(X_i) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X_i)du, \quad S_n^c = \text{ comonotonic sum}$$

Bernard, Rü, Vanduffel (2013), Puccetti, Rü (2012),
Wang, Wang (2011)

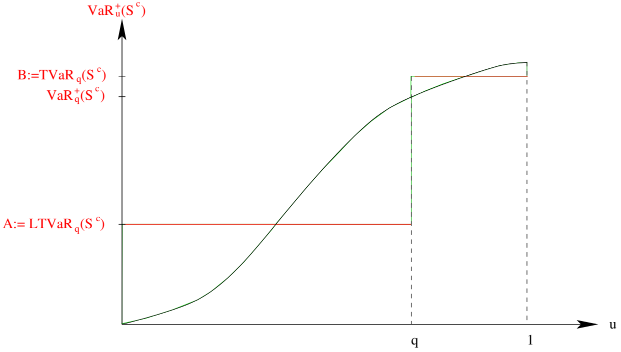
Embrechts, Puccetti (2006), Embrechts, Puccetti, Rü (2013),
Puccetti, Rü (2013)

$$\overline{\mathrm{VaR}}_\alpha(S_n) \sim \mathrm{TVaR}_\alpha(S_n^c), \quad n \to \infty$$

and $$\underline{\mathrm{VaR}}_\alpha(S_n) \sim \mathrm{LTVaR}_\alpha(S_n^c), \quad n \to \infty$$

Puccetti, Rü (2012), Puccetti, Wang (2013); Wang, Wang (2014); Embrechts, Wang, Wang (2015)

note: mixing (= negative dependence) in upper domain allows to increase VaR upper bound



Rearrangement = Dependence

Theorem (Rü (1983))

Let $\mathfrak{F}(F_1, \dots, F_d)$ *be the set of all joint dfs on* $\mathbb{R}^d$ *with marginals* $F_1, \dots, F_d$.
Let U *be a random variable with* $F_U = U(0,1)$. *Then:*

$$\mathfrak{F}(F_1, \dots, F_d) = \left\{ F_{(f_1(U), \dots, f_d(U))};\ f_i \sim_r F_i^{-1},\ 1 \le i \le d \right\}.$$

$$M(s) = \sup \left\{ P\left(\sum_{i=1}^n L_i \ge s \right);\ L_i \sim F_i \right\}$$
$$= 1 - \inf \left\{ \alpha\,;\ \exists\ f_j^\alpha \sim_r F_j^{-1}\big|_{[\alpha,1]},\ \sum_{j=1}^n f_j^\alpha \ge s \right\}$$

$\rightarrow$ RA-algorithm, precise determination of VaR bounds

Rearrangement algorithm

Puccetti, Rü (2012), Embrechts, Puccetti, Rü (2013)

Pareto(2) marginals and $\alpha = 0.99$

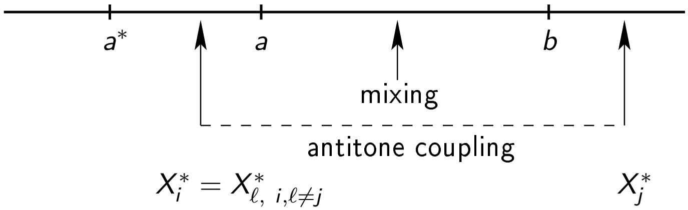
	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

- Any marginal distribution can be approximated by a discrete model with N equally weighted points.
- In order to find $\overline{\mathrm{VaR}}_\alpha$ ($\underline{\mathrm{VaR}}_\alpha$) it is sufficient to consider only the upper (lower) $(1-\alpha)$ part of the support of each marginal distribution.
- For N large enough, it is possible to approximate any dependence between the discrete marginals by a proper rearrangement of the columns of X.
- The rearrangement algorithm (RA) finds efficiently the rearrangement yielding an approximation of $\overline{\mathrm{VaR}}_\alpha$.

- RA calculates numerically $\overline{\mathrm{VaR}}_\alpha$ and $\underline{\mathrm{VaR}}_\alpha$.
- RA can be used with any set of inhomogeneous marginals, with dimensions d up in the several hundreds and for any quantile level α.
- RA provides very accurate estimates ($N = 10^5$).
- RA can be used also with further aggregating functions such as $\times$, min, max, $(\sum x_i - K)_+$, $\psi(x_1 \ldots, x_n)$ supermodular.

Optimal coupling yields (in example) a dependence in which:

- either all (three) rvs are close to each other and sum up to something very close to the minimal sum (→ complete mixability).
- or one of the components is large and the other (two) are small.
- theoretical result: **structure of optimal coupling**



Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

$$\overline{\mathrm{VaR}}_\alpha = 45.9994$$

Optimal Coupling!

	1	2	3	Σ
1	12.28422	21.36068	11.24745	44.8924
2	9.95445	30.62278	9.74172	50.3190
3	21.36068	11.90994	11.56562	44.8362
4	15.51446	14.19109	14.81139	44.5169
5	19.70197	12.28422	13.14214	45.1283
6	17.25742	13.63850	13.63850	44.5344
7	53.77226	9.00000	9.17095	71.9432
8	10.42080	26.38613	10.18034	46.9873
9	13.14214	13.14214	18.36492	44.6492
10	9.17095	53.77226	9.00000	71.9432
11	13.63850	11.24745	19.70197	44.5879
12	18.36492	14.81139	11.90994	45.0862
13	12.69306	19.70197	12.69306	45.0881
14	9.74172	9.95445	30.62278	50.3190
15	10.95229	10.67748	23.49490	45.1247
16	30.62278	9.74172	9.95445	50.3190
17	10.67748	23.49490	10.95229	45.1247
18	14.81139	15.51446	14.19109	44.5169
19	11.56562	17.25742	16.32051	45.1435
20	16.32051	12.69306	15.51446	44.5280
21	37.72983	9.54093	9.35098	56.6217
22	23.49490	10.95229	10.67748	45.1247
23	9.00000	9.17095	53.77226	71.9432
24	10.18034	10.42080	26.38613	46.9873
25	14.19109	18.36492	12.28422	44.8402
26	26.38613	10.18034	10.42080	46.9873
27	9.54093	9.35098	37.72983	56.6217
28	11.24745	16.32051	17.25742	44.8254
29	11.90994	11.56562	21.36068	44.8362
30	9.35098	37.72983	9.54093	56.6217
Σ	494.99920	494.99920	494.99920	NA

$$N = 10^5 \;\Rightarrow\; \overline{\mathrm{VaR}}_\alpha = 45.99$$

(A)	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.10153	9.10153	9.10153	27.3046
3	9.20621	9.20621	9.20621	27.6186
4	9.31421	9.31421	9.31421	27.9426
5	9.42572	9.42572	9.42572	28.2772
6	9.54093	9.54093	9.54093	28.6228
7	9.66004	9.66004	9.66004	28.9801
8	9.78328	9.78328	9.78328	29.3498
9	9.91089	9.91089	9.91089	29.7327
10	10.04315	10.04315	10.04315	30.1295
11	10.18034	10.18034	10.18034	30.5410
12	10.32277	10.32277	10.32277	30.9683
13	10.47079	10.47079	10.47079	31.4124
14	10.62476	10.62476	10.62476	31.8743
15	10.78511	10.78511	10.78511	32.3553
16	10.95229	10.95229	10.95229	32.8569
17	11.12678	11.12678	11.12678	33.3803
18	11.30915	11.30915	11.30915	33.9274
19	11.50000	11.50000	11.50000	34.5000
20	11.70001	11.70001	11.70001	35.1000
21	11.90994	11.90994	11.90994	35.7298
22	12.13064	12.13064	12.13064	36.3919
23	12.36306	12.36306	12.36306	37.0892
24	12.60828	12.60828	12.60828	37.8248
25	12.86750	12.86750	12.86750	38.6025
26	13.14214	13.14214	13.14214	39.4264
27	13.43376	13.43376	13.43376	40.3013
28	13.74420	13.74420	13.74420	41.2326
29	14.07557	14.07557	14.07557	42.2267
30	14.43033	14.43033	14.43033	43.2910
31	14.81139	14.81139	14.81139	44.4342
32	15.22214	15.22214	15.22214	45.6664
33	15.66667	15.66667	15.66667	47.0000
34	16.14986	16.14986	16.14986	48.4496
35	16.67767	16.67767	16.67767	50.0330
36	17.25742	17.25742	17.25742	51.7723
37	17.89822	17.89822	17.89822	53.6947
38	18.61161	18.61161	18.61161	55.8348
39	19.41241	19.41241	19.41241	58.2372
40	20.32007	20.32007	20.32007	60.9602
41	21.36068	21.36068	21.36068	64.0820
42	22.57023	22.57023	22.57023	67.7107
43	24.00000	24.00000	24.00000	72.0000
44	25.72612	25.72612	25.72612	77.1784
45	27.86751	27.86751	27.86751	83.6025
46	30.62278	30.62278	30.62278	91.8683
47	34.35534	34.35534	34.35534	103.0660
48	39.82483	39.82483	39.82483	119.4745
49	49.00000	49.00000	49.00000	147.0000
50	69.71068	69.71068	69.71068	209.1320
Σ	851.72901	851.72901	851.72901	

(B)	1	2	3	Σ
1	13.43376	15.66667	15.66667	44.7671
2	12.86750	14.81139	17.25742	44.9363
3	15.66667	13.14214	16.14986	44.9587
4	14.43033	16.14986	14.43033	45.0105
5	15.22214	11.90994	17.89822	45.0303
6	13.74420	17.89822	13.43376	45.0762
7	19.41241	12.86750	12.86750	45.1474
8	20.32007	11.70001	13.14214	45.1622
9	18.61161	14.43033	12.13064	45.1726
10	13.14214	13.43376	18.61161	45.1875
11	16.14986	12.36306	16.67767	45.1906
12	11.12678	13.74420	20.32007	45.1910
13	11.70001	12.13064	21.36068	45.1913
14	11.50000	22.57023	11.12678	45.1970
15	21.36068	11.50000	12.36306	45.2237
16	14.81139	15.22214	15.22214	45.2557
17	12.60828	21.36068	11.30915	45.2781
18	14.07557	18.61161	12.60828	45.2955
19	17.89822	12.60828	14.81139	45.3179
20	11.30915	20.32007	13.74420	45.3734
21	11.90994	14.07557	19.41241	45.3979
22	17.25742	16.67767	11.50000	45.4351
23	12.13064	19.41241	14.07557	45.6186
24	16.67767	17.25742	11.70001	45.6351
25	12.36306	10.78511	22.57023	45.7184
26	10.78511	10.95229	24.00000	45.7374
27	22.57023	11.30915	11.90994	45.7893
28	10.95229	24.00000	10.95229	45.9046
29	24.00000	11.12678	10.78511	45.9119
30	25.72612	10.62476	10.47079	46.8217
31	10.62476	10.47079	25.72612	46.8217
32	10.47079	25.72612	10.62476	46.8217
33	10.32277	27.86751	10.18034	48.3706
34	27.86751	10.18034	10.32277	48.3706
35	10.18034	10.32277	27.86751	48.3706
36	9.91089	10.04315	30.62278	50.5768
37	10.04315	30.62278	9.91089	50.5768
38	30.62278	9.91089	10.04315	50.5768
39	9.78328	34.35534	9.66004	53.7987
40	34.35534	9.66004	9.78328	53.7987
41	9.66004	9.78328	34.35534	53.7987
42	9.42572	9.54093	39.82483	58.7915
43	39.82483	9.42572	9.54093	58.7915
44	9.54093	39.82483	9.42572	58.7915
45	49.00000	9.31421	9.20621	67.5204
46	9.31421	9.20621	49.00000	67.5204
47	9.20621	49.00000	9.31421	67.5204
48	9.00000	9.10153	69.71068	87.8122
49	69.71068	9.00000	9.10153	87.8122
50	9.10153	69.71068	9.00000	87.8122
Σ	851.72901	851.72901	851.72901	

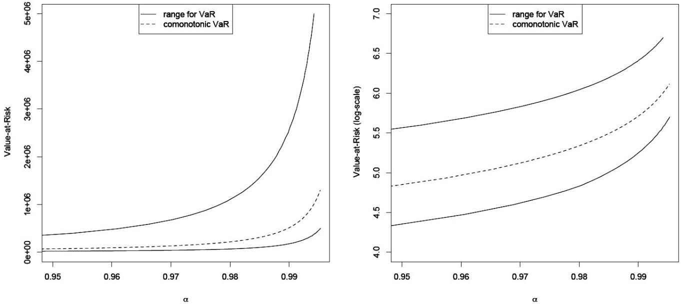
(A): The matrix $\underline{X}^{\alpha}$ defined in (18) for $\alpha = 0.99$ and $N = 50$ (representing comonotonicity among the discrete marginals).

(B): The matrix $\underline{X}^{*}$ derived as an output of the iterative rearrangement of the columns of $\underline{X}^{\alpha}$. The rows of $\underline{X}^{*}$ are ordered accordingly to their sums. In this example we consider a discretization of $d = 3$ Pareto(2)-distributed risks.

Dependence Uncertainty

$d = 8$	$N = 1.0e05$	*avg time*: 30 secs		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	9.00 – 9.00	72.00	141.67	141.66–141.67
0.995	13.13 – 13.14	105.14	203.66	203.65–203.66
0.999	30.47 – 30.62	244.98	465.29	465.28–465.30
$d = 56$	$N = 1.0e05$	*avg time*: 9 mins		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	45.82 – 45.82	504	1053.96	1053.80–1054.11
0.995	48.60 – 48.61	735.96	1513.71	1513.49–1513.93
0.999	52.56 – 52.58	1714.88	3453.99	3453.49–3454.48
$d = 648$	$N = 5.0e04$	*avg time*: 8 hrs		
α	$\underline{\mathrm{VaR}}_\alpha(L)$ (RA range)	$\mathrm{VaR}^+_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (exact)	$\overline{\mathrm{VaR}}_\alpha(L)$ (RA range)
0.99	530.12 – 530.24	5832.00	12302.00	12269.74-12354.00
0.995	562.33 – 562.50	8516.10	17666.06	17620.45-17739.60
0.999	608.08 – 608.47	19843.56	40303.48	40201.48-40467.92

Estimates for $\overline{\mathrm{VaR}}_\alpha(L)$ and $\underline{\mathrm{VaR}}_\alpha(L)$ for random vectors of Pareto(2)-distributed risks.



VaR range (5), and comonotonic VaR(8) (in log-scale on the right) for the sum of $d = 8$ GPD risks with parameters following Moscadelli (2004), based on RA for $N = 1:0e05$.

How to reduce risk bounds by using partial dependence information?

- higher order marginals (reduced bounds)
- positive, negative dependence restrictions (improved standard bounds)
- information on variance of S_n, correlations of X_i, X_j
- partial information on risk factors (partially specified risk factor models)
- models with subgroup structure

intuition:

- positive dependence information allows to increase lower risk bounds (but not upper bounds)
- negative dependence information allows to decrease upper risk bounds (but not lower risk bounds)

2. Higher dimensional marginals,reduced bounds

$$\mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_J; J \in \mathcal{E}) \subset \mathcal{F}(F_1, \dots, F_n)$$

$F_J = F_{X_J}, \quad X_J = (X_j)_{j \in J}$ for $J \in \mathcal{E}$, $\bigcup_{J \in \mathcal{E}} J = \{1, \dots, n\}$

$\mathcal{F}_{\mathcal{E}}$ **generalized Fréchet class**

$\mathcal{E} = \{\{1\}, \dots, \{n\}\} \Rightarrow \mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_1, \dots, F_n)$

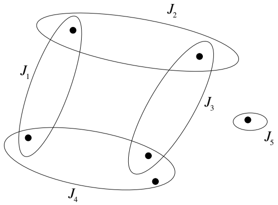
simple marginal class

$\mathcal{E} = \{\{j, j+1\}, 1 \le j \le n-1\} \rightarrow \mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_{1,2}, F_{2,3}, \dots, F_{n-1,n})$

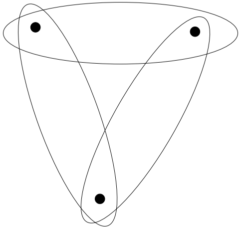
series system

$\mathcal{E} = \{\{1, j\}, 2 \le j \le n\} \rightarrow \mathcal{F}(F_{1,2}, F_{1,3}, \dots, F_{1,n})$

starlike system



$$
\begin{cases} M_{\mathcal{E}}(s) = \sup\{P(X_1 + \cdots + X_n \geq s); F_X \in \mathcal{F}_{\mathcal{E}}\} \\ m_{\mathcal{E}}(s) = \inf\{P(X_1 + \cdots + X_n \geq s); F_X \in \mathcal{F}_{\mathcal{E}}\} \end{cases}
$$



marginal problem: $\mathcal{F}_{\mathcal{E}} \neq \Phi$ (Rü (1991))

decomposable case

duality theorem $\quad M_{\mathcal{E}} \neq \Phi$

$$
\begin{aligned} M_{\mathcal{E}}(\varphi) :&= \sup\left\{\int \varphi dP; P \in M_{\mathcal{E}}\right\} \\ &= \inf\left\{\sum_{J \in \mathcal{E}} \int f_J dP_J; \sum_{J \in \mathcal{E}} f_J \circ \pi_J \geq \varphi\right\}, \quad \varphi \text{ usc} \end{aligned}
$$

Rü (1984), Kellerer (1987)

Bonferoni type bounds

Proposition

$(E_i, \mathcal{A}_i), (P_J, J \in \mathcal{E})$ *marginal system*

1. $M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \leq \min_{J \in \mathcal{E}} P_J(A_J)$
2. $\mathcal{E} = J_2^n = \{(i,j); i,j \leq n\}$,

$$q_i = P_i(A_i^c), \quad q_{ij} = P_{ij}(A_i^c \times A_j^c)$$

$$\begin{cases} M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \leq 1 - \sum q_i + \sum_{i<j} q_{ij} \\ m_{\mathcal{E}}(A_1 \times \cdots \times A_n) \geq 1 - \sum q_i + \sup_{\tau \in T} \sum_{(i,j) \in \tau} q_{ij} \end{cases}$$

$T =$ *spanning trees of* G_n, *Rü (1991)*

Conditional bounds

sharp bounds by conditioning in *some decomposable cases*!

reduced systems

$\mathcal{E} = \{J_1, \dots, J_m\}$

$\eta_i := \#\{J_r \in \mathcal{E}; i \in J_r\}, \quad 1 \le i \le n$

For X risk vector, $F_X \in \mathcal{F}_{\mathcal{E}}$ define:

$$Y_r := \sum_{i \in J_r} \frac{X_i}{\eta_i}, \quad H_r := F_{Y_r}, \quad r = 1, \dots, m$$

$\mathcal{H} = \mathcal{F}(H_1, \dots, H_m)$ Fréchet class

Proposition (reduced bounds)

$\mathcal{F}_{\mathcal{E}} = \emptyset$ consistent marginal system, then for $s \in \mathbb{R}$

$$M_{\mathcal{E}}(s) \le M_{\mathcal{H}}(s) \quad \textit{and}$$
$$m_{\mathcal{E}}(s) \ge m_{\mathcal{H}}(s)$$

Embrechts, Puccetti (2010), Puccetti, Rü (2012)

Remark

1. *generalized weighting schemes*

$$Y_r^\alpha = \sum_{i=1}^n \alpha_i^r X_i, \quad \begin{cases} \alpha_i^r > 0 & \textit{iff } i \in J_r \quad \textit{and} \\ \sum_{r=1}^n \alpha_i^r = 1 \end{cases}$$

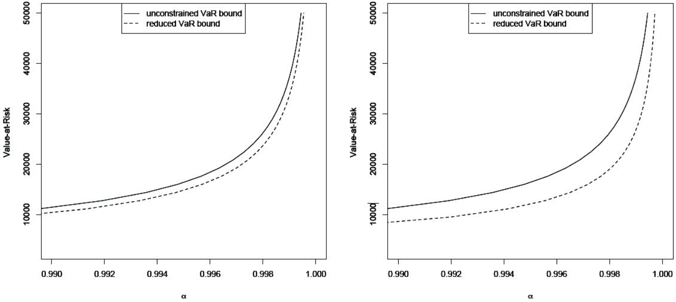
   $\rightarrow$ *parametrized family of bounds*

2. *Rearrangement algorithm can be used to calculate* $M_{\mathcal{H}}$, $m_{\mathcal{H}}$.

Series case $F_{i,i+1}$ 2-dim Pareto

α	$\mathrm{VaR}^+_\alpha(L)$	$\overline{\mathrm{VaR}}^r_\alpha(L)$, (A)	$\overline{\mathrm{VaR}}^r_\alpha(L)$, (B)	$\overline{\mathrm{VaR}}_\alpha(L)$
0.99	5400.00	8496.13	10309.14	11390.00
0.995	7885.28	12015.04	14788.71	16356.42
0.999	18373.67	26832.2	33710.3	37315.70

Estimates for $\mathrm{VaR}_\alpha(L)$ for a random vector of $d = 600$ Pareto(2)-distributed risks under different dependence scenarios: $\mathrm{VaR}^+_\alpha(L)$ ($(L_1, \ldots, L_{600})'$ has copula $C = M$); $\overline{\mathrm{VaR}}^r_\alpha(L)$, (A): the bivariate marginals $F_{2j-1,2j}$ are independent; $\overline{\mathrm{VaR}}^r_\alpha(L)$, (B): the bivariate marginals $F_{2j-1,2j}$ have Pareto copula with $\delta = 1.5$; $\overline{\mathrm{VaR}}_\alpha(L)$: no dependence assumptions are made.



VaR bounds $\overline{\mathrm{VaR}}_\alpha(L)$ (see (5)) and reduced bounds $\overline{\mathrm{VaR}}^r_\alpha(L)$ (see (24a)) for a random vector of $d = 600$ Pareto(2)-distributed risks with fixed bivariate marginals $F_{2j-1,2j}$ generated by a Pareto copula with $\delta = 1.5$, comonotone (left) and by the independence copula (right).

3. Risk bounds with variance and higher order moment constraints

information: $X_i \sim F_i,\ 1 \le i \le n$ and $\mathrm{Var}(S_n) \le s^2 \qquad (*)$

$\to$ partial information on dependence alternatively information on $\mathrm{Cov}(X_i, X_j)$,
Bernard, Rü, Vanduffel (2016)

$$\boxed{\begin{cases} M = \sup\{\mathrm{VaR}_\alpha(S_n); & S_n \text{ satisfies } (*)\} \\ m = \inf\{\mathrm{VaR}_\alpha(S_n); & S_n \text{ satisfies } (*)\} \end{cases}}$$

Theorem

$\alpha \in (0,1)$, $\quad \mathrm{Var}(S_n) \le s^2$, *then*

$$a := \max\left(\mu - s\sqrt{\frac{\alpha}{1-\alpha}}, A\right) \le m \le \mathrm{VaR}_\alpha(S_n) \le M$$

$$\le b := \min\left(\mu + s\sqrt{\frac{\alpha}{1-\alpha}}, B\right), \quad \mu = ES_n$$

Remark

VaR *bounds and convex order worst case dependence structure has relation to convex order minima in upper and lower part*

$$\{S_n \ge \mathrm{VaR}_\alpha(S_n)\} \quad \textit{resp.} \quad \{S_n < \mathrm{VaR}_\alpha(S_n)\}$$

Proposition

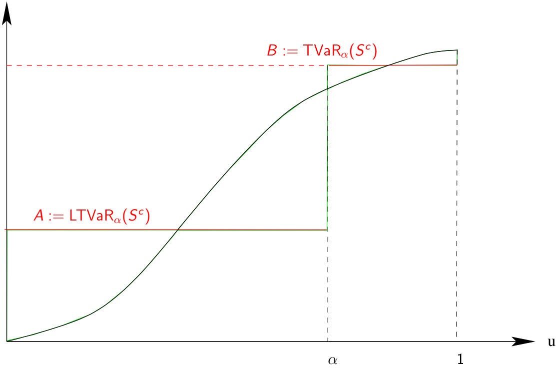
$X_i \sim F_i, \quad F_i^\alpha \sim F_i/[q_i(\alpha), \infty), \quad X_i^\alpha, Y_i^\alpha \sim F_i^\alpha$

a) $$M = \sup_{X_i \sim F_i} \mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) = \sup_{Y_i^\alpha \sim F_i^\alpha} \mathrm{VaR}_0\left(\sum_{i=1}^n Y_i^\alpha\right)$$

b) *If* $S^\alpha = \sum_{i=1}^n Y_i^\alpha \leq_{\mathrm{cx}} \sum_{i=1}^n X_i^\alpha$, *then*

$$\mathrm{VaR}_0\left(\sum_{i=1}^n X_i^\alpha\right) \leq \mathrm{VaR}_0(S^\alpha) = \operatorname{ess\,inf}\left(\sum_{i=1}^n Y_i^\alpha\right) \leq B$$

maximizing VaR $\sim$ maximizing minimal support over all $Y_i \sim F_i^\alpha$ is implied by convex order

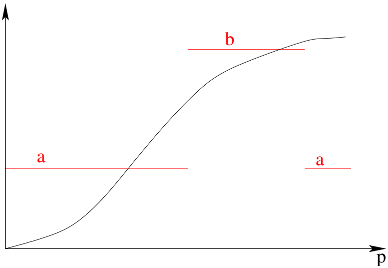


VaR bounds and convex order

Extended Rearrangement Algorithm (ERA)

two alternating steps

1. choice of domain, starting from largest α-domain
2. Rearrangement in upper α-part and in lower $1-\alpha$-part
3. check variance constraint fulfilled
4. shift of domain and iterate



Variation of ERA: Self determined split of domains.

Panel A: Approximate sharp bounds obtained by the ERA

(m_d, M_d)		$n = 10$			$n = 100$		
		$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$	(4.401; 15.72)	(4.091; 21.85)	(3.863; 26.19)	(47.96; 84.72)	(42.48; 188.9)	(39.61; 243.3)
	$\mathrm{VaR}_{99\%}$	(5.486; 28.69)	(4.591; 43.45)	(4.492; 53.22)	(48.99; 129.5)	(46.61; 366.0)	(45.36; 489.5)
	$\mathrm{VaR}_{99.5\%}$	(6.820; 39.48)	(5.471; 59.60)	(4.850; 73.11)	(49.23; 162.8)	(47.54; 499.1)	(46.68; 671.5)

Panel B: Variance-constrained bounds

(a_d, b_d)		$n = 10$			$n = 100$		
		$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$	(4.398; 16.03)	(4.089; 21.92)	(3.861; 26.23)	(47.96; 84.74)	(42.48; 188.9)	(39.61; 243.4)
	$\mathrm{VaR}_{99\%}$	(4.725; 30.20)	(4.589; 43.64)	(4.490; 53.50)	(48.99; 129.6)	(46.59; 367.3)	(45.33; 491.7)
	$\mathrm{VaR}_{99.5\%}$	(4.800; 40.74)	(4.705; 59.80)	(4.634; 73.77)	(49.23; 162.9)	(47.54; 500.0)	(46.65; 676.3)
$d = +\infty$	$\mathrm{VaR}_{95\%}$	(4.372; 16.94)	(4.037; 23.30)	(3.791; 27.96)	(48.01; 87.75)	(42.09; 200.3)	(38.99; 259.2)
	$\mathrm{VaR}_{99\%}$	(4.725; 32.25)	(4.578; 46.77)	(4.470; 57.41)	(49.13; 136.2)	(46.53; 393.1)	(45.18; 527.4)
	$\mathrm{VaR}_{99.5\%}$	(4.806; 43.63)	(4.702; 64.22)	(4.634; 77.72)	(49.39; 172.2)	(47.56; 536.4)	(46.60; 726.9)

Panel C: Unconstrained bounds independent of ϱ

(A_d, B_d)		$n = 10$	$n = 100$
$d = 10{,}000$	$\mathrm{VaR}_{95\%}$,	(3.646; 30.33)	(36.46; 303.3)
	$\mathrm{VaR}_{99\%}$	(4.447; 57.76)	(44.47; 577.6)
	$\mathrm{VaR}_{99.5\%}$	(4.633; 74.11)	(46.33; 741.1)
$d = +\infty$	$\mathrm{VaR}_{95\%}$	(3.647; 30.72)	(36.47; 307.2)
	$\mathrm{VaR}_{99\%}$	(4.448; 59.62)	(44.48; 596.2)
	$\mathrm{VaR}_{99.5\%}$	(4.635; 77.72)	(46.35; 777.2)

Bounds on Value-at-Risk of sums of Pareto distributed risks ($\theta = 3$)

Application to Credit Risk portfolios

asset correlations ϱ^A – default correlations ϱ^D, loans $X_j \sim \mathcal{B}(p)$

example: $n = 10{,}000$, $p = 0.049$ default probability,
$\varrho^D = 0.0157$ (McNeil et al. (2005)),
$s^2 = np(1-p) + n(n-1)p(1-p)\varrho^D$

	(A_d, B_d)	(a_d, b_d)	(m_d, M_d)	KMV	Beta	CreditMetrics
$\mathrm{VaR}_{0.8}$	(0%; 24.50%)	(3.54%; 10.33%)	(3.63%; 10%)	6.84%	6.95%	6.71%
$\mathrm{VaR}_{0.9}$	(0%; 49.00%)	(4.00%; 13.04%)	(4.00%; 13%)	8.51%	8.54%	8.41%
$\mathrm{VaR}_{0.95}$	(0%; 98.00%)	(4.28%; 16.73%)	(4.32%; 16%)	10.10%	10.01%	10.11%
$\mathrm{VaR}_{0.995}$	(4.42%; 100.00%)	(4.71%; 43.18%)	(4.73%; 40%)	15.15%	14.34%	15.87%

The table provides VaR bounds and VaR computed in different models (KMV, Beta, CreditMetrics).

A_d, B_d $\rightarrow$ bounds from marginal information
a_d, b_d $\rightarrow$ bounds with variance constraints

	$p = 0.25\%$			$p = 1\%$		
	(A, B)	(a, b)	KMV	(A, B)	(a, b)	KMV
$\varrho^A = 0\%$	(0%; 50%)	(0.25%; 0.25%)	0.25%	(0.50%; 100%)	(1.00%; 1.00%)	1.0%
$\varrho^A = 6\%$	(0%; 50%)	(0.23%; 3.27%)	1.2%	(0.50%; 100%)	(0.95%; 10.98%)	4.0%
$\varrho^A = 12\%$	(0%; 50%)	(0.23%; 5.05%)	2.1%	(0.50%; 100%)	(0.92%; 16.27%)	6.3%
$\varrho^A = 18\%$	(0%; 50%)	(0.23%; 6.84%)	2.9%	(0.50%; 100%)	(0.90%; 21.18%)	8.7%
$\varrho^A = 24\%$	(0%; 50%)	(0.21%; 8.76%)	3.8%	(0.50%; 100%)	(0.87%; 26.09%)	11.1%
$\varrho^A = 30\%$	(0%; 50%)	(0.20%; 10.85%)	4.8%	(0.50%; 100%)	(0.85%; 31.13%)	13.7%

Unconstrained and constrained upper and lower 0.995-VaR bounds for several combinations of default probability and correlation and VaR in the (one-factor) KMV model

- significant model error, ex. $\varrho^A = 6\,\%$, $p = 0.25\,\%$, then 99.5 % VaR bounds 0.2 %–3.3 %

Higher order moment constraints

Bernard, Rü, Vanduffel, Yao (2017)

$X_i \sim F_i$, $1 \leq i \leq n$ and $ES_n^k \leq c_k$, $k = 2, \ldots, K$

$\rightarrow$ strengthened upper bounds for $\mathrm{VaR}_\alpha(S_n)$, modification of RA-algorithm and theoretical bounds

VaR assessment of a corporate portfolio

	$q =$	KMV	Comon.	Unconstrained	$K=2$	$K=3$	$K=4$
	95%	281.3	393.3	(34.0 ; 2083.3)	(111.8 ; 483.1)	(111.8 ; 433.0)	(111.8 ; 412.8)
$\varrho =$	99%	398.7	2374.1	(56.5 ; 6973.1)	(115.0 ; 943.9)	(117.4 ; 713.3)	(118.2 ; 610.9)
0.05	99.5%	448.5	5088.5	(89.4 ; 10119.9)	(116.9 ; 1285.9)	(118.9 ; 889.5)	(119.8 ; 723.2)
	99.9%	573.1	12905.1	(111.8 ; 14784.9)	(120.2 ; 2718.1)	(121.2 ; 1499.6)	(121.8 ; 1075.9)
	95%	340.6	393.3	(34.0 ; 2083.3)	(97.3 ; 614.8)	(100.9 ; 562.8)	(100.9 ; 560.6)
$\varrho =$	99%	539.4	2374.1	(56.5 ; 6973.1)	(111.8 ; 1245.0)	(115.0 ; 941.2)	(115.9 ; 834.7)
0.10	99.5%	631.5	5088.5	(89.4 ; 10119.9)	(114.9 ; 1709.4)	(117.6 ; 1177.8)	(118.5 ; 989.5)
	99.9%	862.4	12905.1	(111.8 ; 14784.9)	(119.2 ; 3692.3)	(120.8 ; 1995.9)	(121.2 ; 1472.7)
	95%	388.4	393.3	(34.0 ; 2083.3)	(91.5 ; 735.9)	(93.4 ; 697.0)	(92.0 ; 727.9)
$\varrho =$	99%	675.8	2374.1	(56.5 ; 6973.1)	(111.8 ; 1519.5)	(112.4 ; 1174.5)	(113.7 ; 1083.9)
0.15	99.5%	816.1	5088.5	(89.4 ; 10119.9)	(112.8 ; 2098.0)	(115.9 ; 1472.7)	(116.9 ; 1287.6)
	99.9%	1178.4	12905.1	(111.8 ; 14784.9)	(118.4 ; 4531.3)	(120.7 ; 2501.8)	(120.9 ; 1916.6)

We report for various asset correlation levels ϱ and confidence levels q the VaRs under the KMV framework (second column), the comonotonic VaRs (third column) and the VaR bounds in the unconstrained and the constrained case (in the last four columns between brackets – K reflects the number of moments of the portfolio sum that are known). The VaR bounds are obtained using Algorithm 1.

Conclusion:

- impact of variance and higher order moment constraints on VaR bounds
- considerable amount of model risk
- knowledge of marginals + variance (moments) does not always allow to determine VaR's with confidence
- standard risk methods (based on factor models) like KMV, Beta, Credit Metrics report similarly (why? and on what basis?)
- Variance (moment) restriction is a (global) negative dependence assumption; it implies reduction of upper VaR bounds.

4. Improved standard bounds

How does positive/negative dependence information influence risk bounds?

X ***positive upper orthant dependence*** (PUOD)

$$\text{if } \overline{F}_X(x) = P(X > x) \geq \prod_{i=1}^{n} P(X_i > x_i) = \prod_{i=1}^{n} \overline{F}_i(x_i)$$

X ***positive lower orthant dependence*** (PLOD)

$$\text{if } F_X(x) \geq \prod_{i=1}^{n} F_i(x_i), \quad \forall x$$

X POD if X PLOD and PUOD

4.1 One-sided dependence information

$F = F_X$, $\overline{F} = \overline{F}_X$

one-sided dependence information

Let G, H be increasing functions, $F^- \le G$, $H \le F^+$

$$\begin{cases} G \le_{\mathrm{PLOD}} F & \to \textit{positive dependence restriction} \quad \text{(lower tail)} \\ G \le_{\mathrm{PUOD}} F & \to \textit{positive dependence restriction} \quad \text{(upper tail)} \end{cases}$$

example: $G(x) = \prod F_i(x_i)$, X is POD

similarly:

$F \le_{\mathrm{PLOD}} H$, $F \le_{\mathrm{PUOD}} H$ $\to$ *negative dependence restriction*

Williamson, Downs (1990), Denuit, Genest, Marceau (1999), Denuit, Dhaene, Ribas (2001), Embrechts, Höing, Juri (2003), Rü (2005), Embrechts, Puccetti (2006), Puccetti, Rü (2012)

Theorem (improved standard bounds)

X risk vector, marginals $X_i \sim F_i$, $G \uparrow$, $F^- \leq G \leq F^+$, then

a) *If $G \leq_{PLOD} F_X$, then*

$$P\Big(\sum_{i=1}^d X_i \leq s\Big) \geq \bigvee G(s);$$

b) *If $G \geq_{PUOD} F_X$, then*

$$P\Big(\sum_{i=1}^d X_i < s\Big) \leq 1 - \bigvee \overline{G}(s);$$

c) *If G is POD, then*

$$\bigvee\Big(\prod_{i=1}^d F_i\Big)(s) \leq P\Big(\sum_{i=1}^d X_i \leq s\Big),$$

$$P\Big(\sum_{i=1}^d X_i < s\Big) \leq 1 - \bigvee\Big(\prod_{i=1}^d \overline{F}_i\Big)(s).$$

$U(s) := \left\{x \in \mathbb{R}^n; \sum_{i=1}^n x_i = s\right\}$,

$\bigwedge G(s) := \inf_{x \in U(s)} G(x)$ G-infimal convolution,

$\bigvee G(s) := \sup_{x \in U(s)} G(x)$ G-supremal convolution

Improved Fréchet bounds:

- higher dimensional marginals
  various types of Bonferroni bounds
- parameter uncertainty
- 'known domains'

$$F(x) = \Gamma(x), \quad x \in S$$

$$(\text{ or "}\leq\text{" or "}\geq\text{"})$$

$d = 2$ Rachev, Rü (1994), Nelsen, Quesada-Molina, Rodríguez-Lallena, Úbeda-Flores (2001, 2004), Tankov (2011)

$d \geq 2$ Puccetti, Rü, Manko (2016), Lux, Papapantoleon (2016)

digital options on default times for bounds

result: improved VaR-bounds for options

Model for lower bounds

Bignozzi, Puccetti, Rü (2014)

$X = (X_1, \dots, X_d)$ risk vector, $F_i = F_{X_i}$

$\{1, \dots, d\} = \bigcup_{j=1}^{k} I_j$ k-subgroups

$Y = (Y_1, \dots, Y_d)$ satisfies:

$$F_Y(x) = \prod_{j=1}^{k} \min_{i \in I_j} G_j(x_i)$$

i.e. – Y has k independent, homogenous subgroups
– components within subgroups comonotonic

Assumption: $(*)$ $Y \leq X$, positive dependence restriction

where $\leq$ is $\leq_{uo}$ or $\leq_{lo}$, typically: $F_i = G_j$ for $i \in I_j$

If $k = d$ and $F_j = G_j$ then $(*) \sim$ to PUOD resp. PLOD of X

$k = 1$ and $F_i = G_j \Rightarrow X$ comonotonic

Theorem (analytical bounds)

If $Y \leq_{lo} X$, $S = \sum_{i=1}^{d} X_i$, *then*

$$\mathrm{VaR}_\alpha(S) \leq \inf_{\mathfrak{A}_\alpha} \sum_{j=1}^{k} n_j G_j^{-1}(u_j), \quad n_j = |I_j|$$

$\mathfrak{A}_\alpha = \left\{ u \in [\alpha, 1]^k;\ \prod_{j=1}^{k} u_j = \alpha \right\}$ *similar lower bounds.*

simplified formula: Puccetti, Rü, Manko (2016)

Example: Pareto portfolio

lower bounds, homogeneus portfolio, d Pareto(2) risks, k subgroups, d/k variables in each subgroup

$d = 8$	$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$
$\alpha = 0.990$	9.00	72.00	9.00	36.00	9.00	18.00	9.00	9.00
$\alpha = 0.995$	13.14	105.14	13.14	52.57	13.14	26.28	13.14	13.14
$\alpha = 0.999$	30.62	244.98	30.62	122.49	30.62	61.25	30.62	30.62

lower bounds, inhomogenous portfolio, $d/2$ Exp(2) risks and $d/2$ Exp(4) risks

$d = 8$	$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$
$\alpha = 0.990$	2.30	13.82	2.30	9.21	2.30	4.61	2.30	2.30
$\alpha = 0.995$	2.65	15.89	2.65	10.60	2.65	5.30	2.65	2.65
$\alpha = 0.999$	3.45	20.72	3.45	13.82	3.45	6.91	3.45	3.45

essential improvement of lower bounds for $k = 1, 2, 4$;
POD alone does not improve lower bounds

Stronger positive/negative dependence conditions

$X = (X_1, \dots, X_n)$ (sequentially) ***positive cumulative dependent*** (PCD) if

$$P\left(\sum_{i=1}^{k-1} X_i > t_1 \mid X_k > t_2\right) \geq P\left(\sum_{i=1}^{k-1} X_i > t_1\right), \quad 2 \leq k \leq n$$

modification of PCD in Denuit, Dhaene, Ribas (2001)

(sequent.) negative cumulative dependent (NCD) if "$\leq$"

Proposition

a) *If X is PCD, then*

$$S_n^{\perp} = \sum_{i=1}^{n} X_i^{\perp} \leq_{\mathrm{cx}} S_n \leq_{\mathrm{cx}} S_n^c = \sum_{i=1}^{n} X_i^c$$

b) *If X is NCD, then* $S_n \leq_{\mathrm{cx}} S_n^{\perp}$

Consequence:

Corollary (positive dependence restriction)

If X is PCD, then

a) $\mathrm{TVaR}_\alpha(S_n^{\perp}) \leq \mathrm{TVaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^c)$

b) $\underline{\mathrm{LTVaR}_\alpha(S_n^{\perp})} \leq \mathrm{LTVaR}_\alpha(S_n) \leq \mathrm{VaR}_\alpha(S_n) \leq \mathrm{TVaR}_\alpha(S_n^c)$

positive dependence information → improved lower bounds for VaR and TVaR.

Proposition (negative dependence restriction)

If X is NCD, then

a) $S_n \le_{\mathrm{cx}} S_n^{\perp}$ *and*

b) $\mathrm{VaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n^{\perp})$

negative dependence $\to$ improved upper risk bounds

Remark

a) *Modification with negative depencence of sums of blocks*

b) *PCD is not directly comparable to POD, POD does not imply convex ordering of sum*

c) *A stronger ordering wcs = weak conditionally ordered in sequence; Rü (2004)*

$$X \le_{\mathrm{wcs}} Y \Rightarrow \sum_{i=1}^n X_i \le_{\mathrm{cx}} \sum_{i=1}^n Y_i$$

This allows to extend to more general upper resp. lower restrictions. In particular $\le_{\mathrm{WAS}} \; \Rightarrow$ PCD.

Example

Expected shortfall bounds , $Y \leq_{wcs} X$
($d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2))

$d = 8$	unconstrained			$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{ES}}_\alpha$	$\overline{\mathrm{ES}}_\alpha$	DU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S
$\alpha = 0.990$	12.00	38.27	26.27	38.27	−100%	29.15	−65.3%	23.29	−43.0%	19.56	−28.8%
$\alpha = 0.995$	12.00	41.64	29.64	41.64	−100%	31.15	−64.6%	24.52	−42.2%	20.33	−28.1%
$\alpha = 0.999$	12.00	49.27	37.27	49.27	−100%	35.63	−63.4%	27.21	−40.8%	22.02	−26.9%

positive dependence, improvement of lower bounds

$$\text{DU-S} = \overline{\mathrm{VaR}}_\alpha - \underline{\mathrm{VaR}}_\alpha,$$

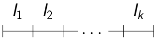
ΔDU-S = reduction of DU-Spread by positive dependence

4.2 (Partial) independence structures

Puccetti, Rü, Small, Vanduffel (2014)

Assumption I)

a) independent subgroups $I_1, \dots, I_k$



b) any dependence within subgroups

$$S = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{i,j}\,, \quad Y_i = \sum_{j=1}^{n_i} X_{i,j} \qquad \text{independent}$$

$$S^{c,k} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} F_{i,j}^{-1}(U_i)$$

Theorem

Under independence assumption I)

$$a^I := LTVaR_\alpha(S^{c,k}) \le \underline{\mathrm{VaR}}_\alpha^I \le \overline{\mathrm{VaR}}_\alpha^I$$
$$\le b^I := TVaR_\alpha(S^{c,k})\,.$$

Gamma distributed groups:

$d = 8$			$k = 1$		$k = 2$		$k = 4$	
	VaR^+_α	$\overline{\mathrm{VaR}}_\alpha$	b^l	e_α	b^l	e_α	b^l	e_α
$\alpha = 0.990$	33.37	38.26	38.27	–	29.15	−23.8%	23.29	−39.1%
$\alpha = 0.995$	36.82	41.63	41.63	–	31.15	−25.2%	24.52	−41.1%
$\alpha = 0.999$	44.59	49.27	49.27	–	35.63	−27.7%	27.21	−44.8%

$d = 8$, 4 Gamma(2,1/2), 4 Gamma(4,1/2), $e_\alpha = 1 - \frac{b^l - a^l}{\overline{\mathrm{VaR}}_\alpha - \underline{\mathrm{VaR}}_\alpha}$.

Pareto distributed groups:

$(a^l; b^l)$	$k = 1$	$k = 2$	$k = 5$	$k = 10$	$k = 25$	$k = 50$
$\alpha = 0.95$	(18.23; 153.72)	(20.21; 116.32)	(22.03; 81.54)	(22.95; 63.93)	(23.76; 48.57)	(24.15; 41.09)
$\alpha = 0.99$	(22.24; 297.84)	(23.14; 208.2)	(23.92; 132.28)	(24.28; 95.97)	(24.59; 65.87)	(24.73; 51.98)
$\alpha = 0.995$	(23.17; 388.91)	(23.8; 269.08)	(24.31; 163.37)	(24.55; 115.34)	(24.74; 76.06)	(24.83; 58.25)
$(\underline{\mathrm{VaR}}_\alpha; \overline{\mathrm{VaR}}_\alpha)$						
$\alpha = 0.95$	(18.24; 153.3)					
$\alpha = 0.99$	(22.26; 297.64)					
$\alpha = 0.995$	(23.2; 388)					

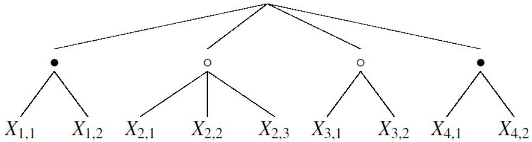
Monte Carlo simulation of marginal and independence bounds, Pareto case with $d = 50$, $\theta_i = \theta = 3$ and $c_i = 1$ for $i = 1, \dots, k$.

$(\underline{e}^\alpha, \overline{e}^\alpha)$	$k = 1$	$k = 2$	$k = 5$	$k = 10$	$k = 25$	$k = 50$
$\alpha = 0.95$	$(-0.05; -0.27)$	$(10.8; 24.12)$	$(20.78; 46.81)$	$(25.82; 58.3)$	$(30.26; 68.32)$	$(32.4; 73.2)$
$\alpha = 0.99$	$(-0.09; -0.07)$	$(3.95; 30.05)$	$(7.46; 55.56)$	$(9.07; 67.76)$	$(10.47; 77.87)$	$(11.1; 82.54)$
$\alpha = 0.995$	$(-0.13; -0.23)$	$(2.59; 30.65)$	$(4.78; 57.89)$	$(5.82; 70.27)$	$(6.64; 80.4)$	$(7.03; 84.99)$

Monte Carlo simulation of marginal and independence bounds, Pareto case with $d = 50$, $\theta_i = \theta = 3$ and $c_i = 1$ for $i = 1, \dots, k$, $\overline{e}^\alpha = \frac{\overline{\mathrm{VaR}}_\alpha - b^I}{\overline{\mathrm{VaR}}_\alpha}$.

Partial independent substructures:

$$\{1, \dots, n\} = \bigcup_{j=1}^{k} I_j, \ (X_{I_j}) \text{ independent for } j \in H \subset \{1, \dots, k\}$$



Partial independent substructures.

Theorem (partial independent substructures)

For $\alpha \in (0,1)$ the following VaR *bounds hold:*

$$a^p = a^p(\alpha, H) := \sum_{i \in \{1,\dots,k\} \setminus H} \mathrm{LTVaR}(S_i^c) + \mathrm{LTVaR}\Big(\sum_{i \in H} S_i^c\Big)$$

$$\leq \mathrm{VaR}(S_d) \leq \sum_{i \in \{1,\dots,k\} \setminus H} \mathrm{TVaR}(S_i^c) + \mathrm{TVaR}\Big(\sum_{i \in H} S_i^c\Big)$$

$$=: b^p(\alpha, H) = b^p.$$

$\sum_{i \in H} S_i^c$ *is an independent sum,*

$\mathrm{TVaR}(S_i^c) = \sum_{j=1}^{n_i} \mathrm{TVaR}(X_{ij})$ *and* $\mathrm{LTVaR}(S_i^c) = \sum_{j=1}^{n_i} \mathrm{LTVaR}(X_{ij})$

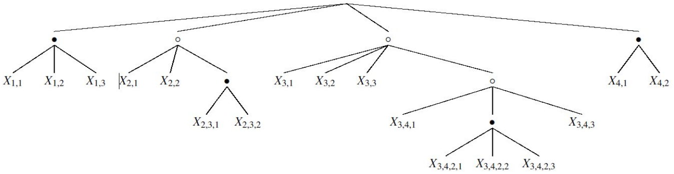
are simple to calculate.

	$\alpha = 0.95$ $F_i \sim \mathrm{Gamma}(\kappa_i^{(1)}, 1)$	$\alpha = 0.995$ $F_i \sim N(\mu_i, 1)$	$\alpha = 0.995$ $F_i \sim N(0,1)$
$(a^I; b^I)$	$(27.58; 76.02)$	$(149.67; 214.67)$	$(-0.33; 64.66)$
$H = \{2,3,4,5\}$	$(26.83; 90.4)$	$(149.57; 236.76)$	$(-0.44; 86.76)$
$H = \{3,4,5\}$	$(25.85; 108.7)$	$(149.47; 257.93)$	$(-0.55; 107.93)$
$H = \{4,5\}$	$(24.8; 128.81)$	$(149.36; 277.66)$	$(-0.64; 127.66)$
$H = \{5\}$	$(23.75; 148.66)$	$(149.28; 294.6)$	$(-0.73; 144.60)$
$(\underline{\mathrm{VaR}}_\alpha\ ;\ \overline{\mathrm{VaR}}_\alpha\ )$	$(23.76; 148.63)$	$(149.29; 294.59)$	$(-0.71; 144.59)$

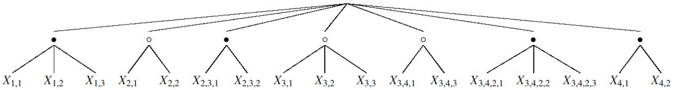
Partial independence bounds with variation of independent substructure, $d = 50$, $k = 5$, $\mu_i = i$.

Remark

a) partial independent graph structures

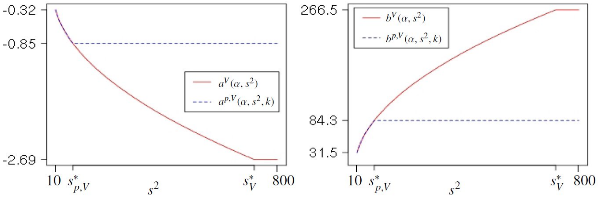


Partial independent graph structures.



Partial independent reduction.

b) combination with variance bounds



(a) $a^{p,V}(\alpha, s^2, k)$ und $a^V(\alpha, s^2)$ as functions of s^2

(b) $b^{p,V}(\alpha, s^2, k)$ and $b^V(\alpha, s^2)$ as functions of s^2

Variance constrained versus independence + variance constrained bounds a^V, $a^{p,V}$ resp. b^V, $b^{p,V}$.

		$d = 10$	$d = 100$
s_V^*	$\alpha = 0.95$	22.39	2239.26
	$\alpha = 0.99$	7.17	717.49
	$\alpha = 0.995$	4.20	420.27

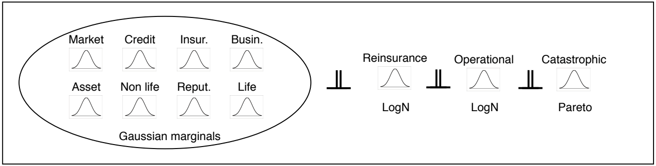
Approximations of critical value s_V^* by Monte Carlo simulation with 10^2 repetitions of 10^5 simulations.

		$d = 100,\ k = 10$				
		$s^2 = 20$	$s^2 = 50$	$s^2 = 100$	$s^2 = 200$	$s^2 = 500$
$(a^{p,V}; b^{p,V})$	$\alpha = 0.95$	$(-1.03; 19.49)$	$(-1.62; 30.82)$	$(-2.29; 43.59)$	$(-3.24; 61.64)$	$(-3.43; 65.23)$
	$\alpha = 0.99$	$(-0.45; 44.5)$	$(-0.71; 70.36)$	$(-0.85; 84.28)$	$(-0.85; 84.28)$	$(-0.86; 84.28)$
	$\alpha = 0.995$	$(-0.32; 63.09)$	$(-0.46; 91.45)$	$(-0.46; 91.45)$	$(-0.45; 91.45)$	$(-0.46; 91.45)$
$(a^V; b^V)$	$\alpha = 0.95$	$(-1.03; 19.49)$	$(-1.62; 30.82)$	$(-2.29; 43.59)$	$(-3.24; 61.64)$	$(-5.13; 97.47)$
	$\alpha = 0.99$	$(-0.45; 44.5)$	$(-0.71; 70.36)$	$(-1.01; 99.5)$	$(-1.42; 140.71)$	$(-2.25; 222.49)$
	$\alpha = 0.995$	$(-0.32; 63.09)$	$(-0.5; 99.75)$	$(-0.71; 141.07)$	$(-1; 199.5)$	$(-1.45; 289.2)$

Approximation of $(a^{p,V}, b^{p,V})$ by Monte Carlo simulation with 10^2 iterations of 10^5 simulations.

Examples (application to insurance portfolio)

$d = 11,\ k = 4$

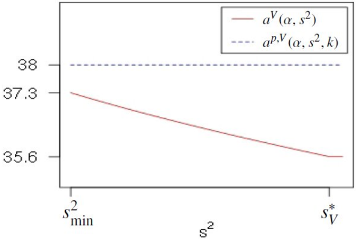


Insurance risk portfolio.

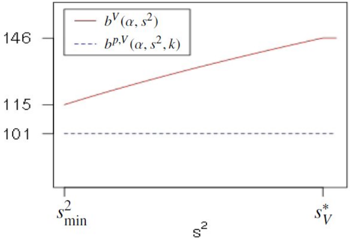
	b^l	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$b^l/\overline{\mathrm{VaR}}_\alpha - 1$
$\alpha = 99\%$	$147.34 - 148.46 - 149.66$	168.37	209.59	-29.2%
	b^l	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$\Delta\,\mathrm{VaR}_\alpha(L_t^+)$
$\alpha = 99.5\%$	$173.37 - 175.18 - 176.96$	202.89	249.55	-29.8%
	b^l	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$	$\Delta\,\mathrm{VaR}_\alpha(L_6^+)$
$\alpha = 99.9\%$	$250.41 - 256.04 - 262.47$	304.63	367.70	-30.4%

upper bounds b^l, VaR_α^+ = comonotonic VaR and $\overline{\mathrm{VaR}}_\alpha$ for 11-dimensional insurance portfolio

Comparison of independence and variance bounds



(a) $a^{p,V}(\alpha, s^2, k)$ and $a^V(\alpha, s^2)$ as function of s^2



(b) $b^{p,V}(\alpha, s^2, k)$ and $b^V(\alpha, s^2)$ as function of s^2

4.3 Two-sided improved bounds

improved bounds: positive dependence: $\underline{G} \leq F_X$ or $\overline{F}_X \geq \overline{G}$; or negative dependence

problem: needs strong positive dependence and d small

two-sided bounds: $\underline{Q} \leq C \leq \overline{Q}$, $\quad \underline{Q}, \overline{Q}$ quasi-copulas

result: **two-sided improved bounds**
based on multiset-inclusion exclusion principle

example:
$$\begin{aligned} 1_{B_1 \cup B_2 \cup B_3} = {} & 1_{B_1} + 1_{B_2} + 1_{B_3} \\ & - 1_{B_1 \cap B_2} - 1_{B_2 \cap B_3} - 1_{B_1 \cap B_3} + 1_{B_1 \cap B_2 \cap B_3} \end{aligned}$$

needs upper and lower bounds!
parsimonious representation $\rightarrow$ reduction scheme

Lux, Rü (2018)

Examples

1. $C^*(u) - \delta \leq C \leq C^*(u) + \delta$, C^* Gaussian equi-correlated

	$\varrho = -0.1$			$\varrho = 0.4$			$\varrho = 0.8$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	3.4 : 45.0	8.2 : 24.8	60	3.6 : 41.2	7.2 : 28.1	44	7.8 : 31.4	9.2 : 26.2	28
0.99	9.0 : 106.2	15.9 : 56.7	58	9.0 : 105.3	14.9 : 80.8	32	17.4 : 84.9	18.6 : 82.2	6
0.995	13.3 : 153.0	19.0 : 90.0	49	13.3 : 153.0	18.0 : 153.0	3	23.4 : 126.0	22.8 : 125.0	0

Improved standard bounds on VaR of $X_1 + \ldots + X_5$ and VaR estimates via reduction schemes for $\delta = 0.0005$.

2. $C^{\underline{\Sigma}} \leq C \leq C^{\overline{\Sigma}}$, Gaussian-copula

	$\underline{\varrho} = -0.1$, $\overline{\varrho} = 0.2$			$\underline{\varrho} = 0.3$, $\overline{\varrho} = 0.5$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	3 : 32	8 : 26	38	1 : 30	7 : 29	24
0.99	9 : 74	20 : 52	51	2 : 74	18 : 63	37
0.995	13 : 104	26 : 70	52	3 : 104	25 : 86	40

Improved standard bounds on VaR of $X_1 + \ldots + X_4$ and VaR estimates computed via reduction schemes using $C^{\underline{\Sigma}}$ and $C^{\overline{\Sigma}}$.

3. Subgroup models, $C^{\theta_1} \leq C_m \leq C^{\theta_2}$ bounds for subgroups copulas by Frank-copulas

	$m = 8$			$m = 4$			$m = 2$		
α	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %	i. standard (low : up)	scheme (low : up)	impr. %
0.95	42 : 113	59 : 86	62	22 : 150	39 : 112	43	12 : 193	28 : 150	33
0.99	82 : 210	108 : 147	70	42 : 264	67 : 175	51	21 : 329	42 : 218	43
0.995	105 : 266	135 : 180	72	53 : 329	83 : 206	55	43 : 403	51 : 252	44

Improved standard bounds and VaR estimates via reduction schemes for $X_1 + \ldots + X_{16}$ given distributions of subgroups.

5. Partially specified risk factor models

Bernard, Rü, Vanduffel, Wang (2017)

risk vector $X = (X_1, \dots, X_n)$, risk factor Z

$$\boxed{\textbf{factor model:}\quad X_j = f_j(Z, \varepsilon_j),}$$

Z systemic risk factor, ε_j individual risk factors

Assumption: known $H_j \sim (X_j, Z)$, $1 \le j \le n$

but not joint distribution! $\to$ marginals F_j and $Z \sim G$

$H = (H_j), F = (F_j)$, conditional distribution $F_{j|z}$ known

$A(H) = \{(X, Z); (X_j, Z) \sim H_j,\ 1 \le j \le n\}$

partially specified risk factor model

$$\boxed{\begin{cases} \overline{M}^b(t) = \sup\{P(S \ge t); (X, Z) \in A(H)\} \\ \overline{\mathrm{Var}}_\alpha^b \quad = \sup\{\mathrm{VaR}_\alpha(S); (X, Z) \in A(H)\} \end{cases}}$$

similarly $\overline{\mathrm{VaR}}_\alpha^b$, $\overline{\mathrm{TVaR}}_\alpha^b$, $\underline{\mathrm{VaR}}_\alpha^b$, ...

Proposition (improvement over marginal bounds)

$$\overline{M}^b(t) \le \overline{M}(t) := \sup\{P(S \ge t); X \in A_1(F)\}$$

$\overline{\mathrm{VaR}}_\alpha^b \le \overline{\mathrm{VaR}}_\alpha, \quad \overline{\mathrm{TVaR}}_\alpha^b \le \overline{\mathrm{TVaR}}_\alpha$

Let $F_{j|z} = F_{X_j|Z=z}$, $F_z = (F_{j|z})$

$\overline{M}_z(t) = \sup\left\{P\left(\sum_{j=1}^n X_{j,z} \ge t\right); (X_{j,z})_j \in A_1(F_z)\right\}$

similarly $\underline{M}_z(t)$, $\overline{\mathrm{VaR}}_\alpha(S_z), \dots, S_z = \sum X_{j,z}$

Proposition (sharp tail risk bounds)

We have

a) $\overline{M}^b(t) = \int \overline{M}_z(t) dG(z), \quad \underline{M}_b(t) = \int \underline{M}_z(t) dG(z)$

b) $\overline{\mathrm{VaR}}_\alpha^b = (\overline{M}^b)^{-1}(1-\alpha), \quad \underline{\mathrm{VaR}}_\alpha^b = (\underline{M}^b)^{-1}(1-\alpha)$

$$(\overline{M}^b)^{-1}(1-\alpha) = \sup\{t : \overline{M}^b(t) > 1-\alpha\}$$

Mixture representation:

$X = X_Z$ with $X_z = (X_{j,z}) \in A(F_z)$, $Z \perp (X_{j,z})$.

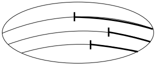
$F_S = \int F_{S_z} dG(z)$

$\alpha \in \Phi$, $b_\alpha := \operatorname{ess\,sup}_{z,G} \mathrm{VaR}_{\alpha(z)}(S_z)$ $\quad \alpha$ defined on range of Z.

Proposition (VaR representation of mixtures)

$$\mathrm{VaR}_\beta(S_Z) = b^* := \inf\left\{ b_\alpha; \alpha \in \Phi, \int \alpha(z) dG(z) \geq \beta \right\}$$

$q_z(\alpha) := \mathrm{VaR}_\alpha(S_z) \uparrow_\alpha$



$Z = z$

$\gamma \in \mathbb{R} : \gamma_z = q_z^{-1}(\gamma) = F_{S_z}(\gamma)$

inverse γ-quantile of $S_z \sim$ probability on $\{Z = z\}$

$$\gamma^*(\beta) := \inf\left\{ \gamma; \int \gamma_z dG(z) \geq \beta \right\},$$

i.e. choose smallest γ such that total probability of test γ_z

$$\int \gamma_z dG(z) \geq \beta.$$

Theorem (worst case VaR in factor model)

a) $\mathrm{VaR}_\beta(S_Z) = \gamma^*(\beta)$

b) $\overline{\mathrm{VaR}}_\beta^b = \overline{\gamma}^*(\beta) = \inf\{\gamma; \int \overline{\gamma}_z dG(z) \geq \beta\}$

$\overline{q}_z(\alpha) = \overline{\mathrm{VaR}}_\alpha(S_z), \overline{\gamma}_z = (\overline{q}_z)^{-1}(\gamma)$

worst case inverse γ-quantile

simplified upper bound:

$t_z(\beta) = \mathrm{TVaR}_\alpha(S_z^c) = \sum_{j=1}^n \mathrm{TVaR}_\alpha(X_{j,z})$

$\Rightarrow q_z(\beta) \leq t_z(\beta)$

$\Rightarrow \overline{\gamma}^*(\beta) \leq \gamma_t^*(\beta) = \inf\left\{\gamma; \int t_z^{-1}(\gamma) dG(z) \geq \beta\right\}$

Corollary

a) $\overline{\mathrm{VaR}}_\alpha^b = \overline{\gamma}^*(\beta) \le \gamma_t^*(\beta)$

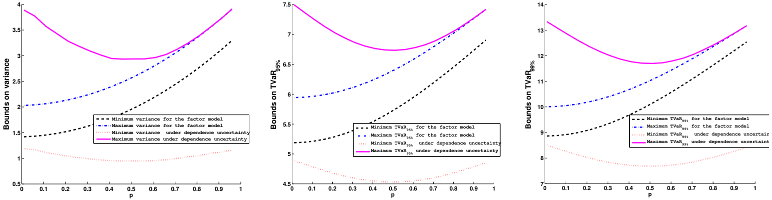
b) $T_z^+ := \mathrm{TVaR}_U(S_z^c)$, $U \sim U(0,1)$, *then*

$$\mathrm{VaR}_\beta(T_Z^+) = \gamma_t^*(\beta)$$

various methods to calculate these bounds

Example (Pareto distributions: p parameter for dependence)

$$\begin{aligned}
X_i^1 &= (1-Z)^{-1/3} - 1 + \varepsilon_i^1 \\
X_i^2 &= I((1-Z)^{-1/3} - 1) + (1-I)(Z^{-1/3} - 1) + \varepsilon_i^2 \\
\varepsilon_i^j &\sim \ \mathrm{Pareto}(\theta_2) \\
\varepsilon_i^1, \varepsilon_i^2 &\sim \ \mathrm{Pareto}(4), \ \ Z \sim U(0,1) \\
I \sim \mathfrak{B}(1,p), &\quad \Delta := 1 - \frac{\mathrm{VaR}_\alpha(T_Z^+) - \mathrm{VaR}_\alpha(T_Z^-)}{\mathrm{TVaR}_\alpha(S^c) - \mathrm{LTVaR}_\alpha(S^c)}
\end{aligned}$$



bounds for the variance, TVaR at 95% and TVaR at 99%

p dependence parameter; $p = 0 \sim$ strong negative dependence; $p = 1 \sim$ strong positive dependence

$n = 50$	VaR_α	$\mathrm{TVaR}_\alpha(S^c)$	$\mathrm{VaR}_\alpha\left(T_Z^+\right)$	$\mathrm{LTVaR}_\alpha(S^c)$	$\mathrm{VaR}_\alpha\left(T_Z^-\right)$	Δ
$p = 0.0$	157	378	266	68	149	62%
$p = 0.2$	158	354	267	69	151	59%
$p = 0.4$	164	340	271	70	157	58%
$p = 0.5$	169	338	274	70	161	58%
$p = 0.6$	175	340	278	70	167	59%
$p = 0.8$	189	354	289	69	181	62%
$p = 1.0$	205	378	300	68	198	67%

upper and lower VaR bounds, $\theta_2 = 4$, VaR_α independence

$p \approx 0 \Rightarrow$ strong negative dependence, $p \approx 1 \Rightarrow$ strong positive dependence

Applications and generalizations

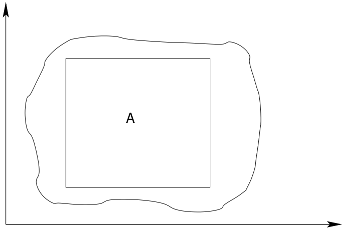
generalized mixture model: $Z \in D = D_1 + D_2 + D_3$

$$P^X = p_1 P^1 + p_2 P^2 + p_3 P^3, \quad p_i = P(Z \in D_i)$$

$z \in D_1 \Rightarrow P_z^1 = P^1$ fixed distribution

$z \in D_2 \Rightarrow P_z^2 \in \mathcal{F}(F_z)$ risk factor information

$z \in D_3 \Rightarrow P_z^3 \in \mathcal{F}((G_j))$ marginal information



special case:
Bernard, Vanduffel (2014)
central part
$\{Z = 0\} = \{X \in A\} \to P^1$
$\{Z = 1\} = \{X \in A^c\}$
only marginal information

Consequence:

a) $$\overline{M}(t) = p_1 P^1\Big(\sum_{j=1}^n X_j \geq t\Big) + \int_{D_2} \overline{M}^b_{2,z}(t)\,dP^Z(z) + p_3\overline{M}_3(t)$$

b) $$S = \sum_{i=1}^n X_i \leq_{\mathrm{cx}} I(Z \in D_1)F_1^{-1}(U) + I(Z\in D_2)S^c_{2,Z} + I(Z\in D_3)S^c_3$$

$$S^c_{2,z} = \textstyle\sum_{j=1}^n F^{-1}_{j|z}(U), \quad S^c_{2,Z} \sim \text{conditionally comonotone}$$

Examples (mixture models)

$X_j = f_j(Z, \varepsilon_j)$

Bernoulli mixture model (credit risk)

$$P(X = x \mid Z = z) = \prod_{i=1}^n p_i(z)^{x_i}(1 - p_i(z))^{1-x_i}$$

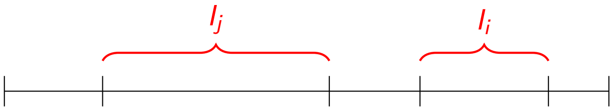
mult. variance mixture model

$$X = \mu + \sqrt{W}\varepsilon, \quad \varepsilon \sim N(0,\Sigma), \quad W \text{ stochastic volatility}$$

6. Subgroup structure models

subgroup models in: Bignozzi, Puccetti, Rü (2015) and Puccetti, Rü, Small, Vanduffel (2015)

subgroups $\{1,\dots,d\} = \bigcup_{i=1}^{k} I_i$, risk vector X



BPR: $\exists\, Z \le X$ positive dependence restriction

(or $X \le Z$ negative dependence restriction)

$\le$ positive dependence ordering

(e.g. $\le_{\mathrm{uo}}, \le_{\mathrm{c}}, \le_{\mathrm{wcs}}, \le_{\mathrm{sm}}, \le_{\mathrm{dcx}}$)

Z independent subgroups, Z_{I_i} comonotonic

PRSV: $\{X_{I_i}\}$ independent, within subgroups any dependence

stochastic ordering within subgroups and between subgroups

Rü, Witting (2017)

X risk vector, Z comparison vector, split into subgroups

$Y_i = \sum_{j \in I_i} X_j, \quad W_i = \sum_{j \in I_i} Z_j$ subgroup sums

$Y_i \sim G_i, \quad W_i \sim H_i, \quad Y = (Y_1, \dots, Y_k), W = (W_1, \dots, W_k)$

$S = \sum_{i=1}^k Y_i, \quad T = \sum_{i=1}^k W_i$

Ordering within subgroups: $G_i \leq H_i$ (resp. $G_i \geq H_i$)

plus **ordering of copulas:** $C_Y \leq C_W$ (resp. $=$ or $\geq$) **between subgroups**

Stochastic Ordering

$X = (X_1, \dots, X_m), \quad Y = (Y_1, \dots, Y_m)$
X **conditional increasing** (CI) if

$$X_i \uparrow_{\mathrm{st}} X_J, \quad \forall\, J \subset \{1, \dots, m\} \setminus \{i\}$$

X **conditional increasing in sequence** (CIS) if

$$X_i \uparrow_{\mathrm{st}} (X_1, \dots, X_{i-1}), \quad \forall\, i \le m$$

$X \le_{\mathrm{wcs}} Y$ **weakly conditional in sequence order** if

$$\mathrm{Cov}(1_{(X_i > x_i)}, f(X_{i+1}, \dots, X_m)) \le \mathrm{Cov}(1_{(Y_i > x_i)}, f(Y_{i+1}, \dots, Y_m)) \quad \text{for all } f \uparrow$$

X **weakly associated in sequence** (WAS) if $X^{\perp} \le_{\mathrm{wcs}} X$

$$\Leftrightarrow P^{X_{(i+1)}} \le_{\mathrm{st}} P^{X_{(i+1)} | X_i > x_i}, \quad \forall\, i, \forall\, x_i,$$

$$X_{(i+1)} = (X_{i+1}, \dots, X_m)$$

Theorem (relations between orderings)

a) $CI \Rightarrow CIS \Rightarrow WA \Rightarrow WAS$

b) $\forall\, i : X_i \stackrel{d}{=} Y_i$ *and* $X \le_{\mathrm{wcs}} Y \Rightarrow X \le_{\mathrm{sm}} Y$

c) $\forall\, i : X_i \le_{\mathrm{cx}} Y_i$ *and* $X \le_{\mathrm{wcs}} Y \Rightarrow X \le_{\mathrm{dcx}} Y$

d) *If* $C_X = C_Y$ *is CI and* $X_i \le_{\mathrm{cx}} Y_i, \forall\, i$ *then* $X \le_{\mathrm{wcs}} Y$

e) $C_X \le_{\mathrm{sm}} C_Y$ *and* C_Y *is CI,*
then $X_i \le_{\mathrm{cx}} Y_i \Rightarrow X \le_{\mathrm{wcs}} Y$

Remark

c), d) implies: $C_X = C_Y$ *is CI,* $X_i \le_{\mathrm{cx}} Y_i$
$\Rightarrow X \le_{\mathrm{dcx}} Y$ *(Müller, Scarsini (2001))*

A) Elliptical Copulas

$S \sim E_d(\mu, \Sigma, \Phi)$ if $\varphi_X(t) = e^{it^\top \mu}\Phi(t^\top \Sigma t)$

$\Rightarrow X \overset{d}{=} \mu + RAU, \quad A^\top A = \Sigma,\ U \sim \text{unif}(S_{d-1})$ and $R \perp U$,

$\Sigma \sim$ correlation matrix of X

$A \in \mathbb{R}^{d \times d}$ ***M*-matrix**, if $a_{ij} \leq 0, \forall\, i \neq j$ and principal minors positive.

Proposition

$X \sim E_d(0, \Sigma, \Phi)$, then X is CI $\Leftrightarrow \Sigma^{-1}$ is an M-matrix

Rü (1981) for multivariate normal

Theorem

$X \sim E_d(\mu_1, \Sigma_1, \Phi)$, $Y \sim E_d(\mu_2, \Sigma_2, \Phi)$

a) $\mu_1 \leq \mu_2$, $\Sigma_1 \leq_{\mathrm{psd}} \Sigma_2 \Rightarrow X \leq_{\mathrm{icx}} Y$

b) $\mu_1 = \mu_2$, $\sigma_{ij}^{(1)} \leq \sigma_{ij}^{(2)}, \forall\, i \neq j$, $\sigma_{ii}^{(1)} = \sigma_{ii}^{(2)}, \forall\, i$, *then* $X \leq_{\mathrm{sm}} Y$

c) $\mu_1 = \mu_2$, $\sigma_{ij}^{(1)} \leq \sigma_{ij}^{(2)}$, $\forall\, i, j$, *then* $X \leq_{\mathrm{dcx}} Y$

a) Pan, Qiu, Hu (2016);

b), c) Block, Sampson (1988);

Müller,Scarsini (2000) normal case ;

Ansari, Rüschendorf (2019) general case

b) Archimedean Copulas

$$C(x_1, \dots, x_d) = \Psi\Big(\sum_{i=1}^{d} \Psi^{-1}(x_i)\Big)$$

Theorem

$C = C_\Psi$, $\Psi \in C^d(\mathbb{R}_+)$, then

a) *C_Ψ is CI $\Leftrightarrow (-1)^{d-1}\Psi^{(d-1)}$ log-convex*

b) *Ψ completely monotone, i.e. d-alternating, $\forall\, d$, then C_Ψ is CI*

Müller, Scarsini (2001, 2005)

Theorem

$C_i = C_{\Phi_i}$ Archimedean copulas, Φ_i completely monotone
If $\Phi_1^{-1} \circ \Phi_2$ completely monotone, then $C_1 \le_{\mathrm{sm}} C_2$

Wei, Hu (2002)

Dependence structures within subgroups

$C = C_Y$ copula between subgroups fixed

Proposition

$C = C_Y$ is WAS (or CIS)

a) *If $Y_i \leq_{\mathrm{cx}} W_i$, $1 \leq i \leq k$, $C_W = C = C_Y$, then*

$$S = \sum_{i=1}^{k} Y_i \leq_{\mathrm{cx}} T = \sum_{i=1}^{k} W_i$$

in particular: $\mathrm{LTVaR}_\alpha(T) \leq \mathrm{VaR}_\alpha(S) \leq \mathrm{TVaR}_\alpha(T)$

b) *If $W_i \leq_{\mathrm{cx}} Y_i$. $1 \leq i \leq k$ and $C_W = C = C_Y$, then*

$$T \leq_{\mathrm{cx}} S \text{ and } \mathrm{TVaR}_\alpha(T) \leq \mathrm{TVaR}_\alpha(S)$$

Remark

a) $X_{I_i} \leq_{\text{wcs}} Z_{I_i}$, $X_j \leq_{\text{cx}} Z_j$

$$\Rightarrow \quad X_{I_i} \leq_{\text{dcx}} Z_{I_i} \text{ and } Y_i \leq_{\text{cx}} W_i$$

Similarly: $X_{I_i} \leq_{\text{sm}} Z_{I_i} \Rightarrow Y_i \leq_{\text{cx}} W_i$

In particular
unknown dependence within subgroups, *then*

$$X_{I_i} \leq_{\text{sm}} Z_{I_i} = (F_j^{-1}(U_i))_{j \in I_i}$$

$$\Rightarrow \quad Y_i \leq_{\text{cx}} W_i = \sum_{j \in I_i} F_j^{-1}(U_i)$$

If $(U_1, \ldots, U_k) \sim C$,

then: $X \leq_{\text{sm}} Z$ *and* $S \leq_{\text{cx}} T$

Remark

b) ***partially specified risk factor models within subgroups***

Bernard, Rü, Vanduffel, Wang (2016)

$X_i = f_j(Z_i^f, \varepsilon_j)$, $j \in I_i$, *partially specified risk factor models*

$$\Rightarrow Y_i = \sum_{j \in I_i} X_j \leq_{\mathrm{cx}} W_i = \sum_{j \in I_i} X^c_{j|Z_i^f}$$

conditionally comonotone

$C = C_W$, $R_z^- = \mathrm{LTVaR}_V(S_z^c)$, $R_z^+ = \mathrm{TVaR}_V(S_z^c)$,
$V \sim U(0,1)$, $z = (z_j)$, *then*

$$\mathrm{VaR}_\alpha(R^-_{Z^f}) \leq \mathrm{VaR}_\alpha(S) \leq \mathrm{VaR}_\alpha(R^+_{Z^f})$$

Example

d risks, k independent subgroups I_i
partially specified risk factor models within subgroups

half of $X_j : X_j = (1 - U_i)^{-1/3} - 1 + \varepsilon_j$

half of $X_j : X_j = p((1 - U_i)^{-1/3} - 1) + (1 - p)(U_i^{-1/3} - 1) + \varepsilon_j$

$\varepsilon_j \sim Pareto(4), p \in (0, 1)$
$C = C^{\perp}$ independent subgroups copula, $C = C_Y = C_W$

	$p = 0.0$	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 1.0$
$(\underline{\mathrm{VaR}}_\alpha, \overline{\mathrm{VaR}}_\alpha)$	(68; 392)	(69; 367)	(70; 349)	(69; 368)	(68; 391)

Sharp VaR bounds with marginal information only $d = 100$, $\alpha = 0.95$.

Example (cont.)

		$p = 0.0$	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 1.0$
$k = 1$	$b(\mathrm{TVaR}_\alpha)$	(68; 474)	(69; 376)	(70; 372)	(69; 384)	(68; 402)
	$b(\mathrm{TVaR}^f_\alpha)$	(72; 297)	(72; 301)	(71; 320)	(69; 351)	(68; 376)
	$b(\mathrm{VaR}^f_\alpha)$	(132; 263)	(134; 265)	(145; 273)	(164; 286)	(182; 296)
$k = 2$	$b(\mathrm{TVaR}_\alpha)$	(72; 385)	(74; 295)	(74; 295)	(74; 301)	(73; 313)
	$b(\mathrm{TVaR}^f_\alpha)$	(76; 231)	(75; 234)	(75; 247)	(74; 269)	(73; 287)
	$b(\mathrm{VaR}^f_\alpha)$	(121; 209)	(122; 210)	(130; 216)	(146; 227)	(158; 237)
$k = 5$	$b(\mathrm{TVaR}_\alpha)$	(77; 305)	(77; 222)	(77; 226)	(77; 229)	(77; 234)
	$b(\mathrm{TVaR}^f_\alpha)$	(79; 173)	(79; 174)	(78; 183)	(77; 197)	(77; 208)
	$b(\mathrm{VaR}^f_\alpha)$	(110; 161)	(110; 162)	(116; 167)	(125; 174)	(133; 180)
$k = 10$	$b(\mathrm{TVaR}_\alpha)$	(79; 266)	(79; 186)	(79; 193)	(79; 193)	(79; 195)
	$b(\mathrm{TVaR}^f_\alpha)$	(80; 144)	(80; 145)	(80; 151)	(79; 161)	(79; 169)
	$b(\mathrm{VaR}^f_\alpha)$	(101; 137)	(102; 138)	(107; 141)	(113; 146)	(119; 151)

VaR bounds with and without factor model information for various group sizes, $d = 100$, $\alpha = 0.95$, $k = 1, 2, 5, 10$.

Dependence structure between subgroups

$C = C_Y, \quad D = C_W$

Proposition

Let C or D be CI

a) $C \leq_{\mathrm{wcs}} D$ *and* $Y_i \leq_{\mathrm{cx}} W_i$, *then*

$$S = \sum_{i=1}^{k} Y_i \leq_{\mathrm{cx}} T = \sum_{i=1}^{k} W_i$$

in particular:

$$\mathrm{LTVaR}_\alpha(T) \leq \mathrm{Var}_\alpha(S) \leq \mathrm{TVaR}_\alpha(S) \leq \mathrm{TVaR}_\alpha(T)$$

b) $W_i \leq_{\mathrm{cx}} Y_i$, $D \leq_{\mathrm{wcs}} C$, *then*

$$T \leq_{\mathrm{cx}} S \text{ and } \mathrm{TVaR}_\alpha(T) \leq \mathrm{TVaR}_\alpha(S).$$

Similar comparison also in terms of $\leq_{\mathrm{sm}}$, $\leq_{\mathrm{dcx}}$

Example

$X = (X_i), \qquad X_i \sim Pareto(3)$

assume: $C \leq_{\text{wcs}} D$

D Gauss- or t-copula and Σ^{-1} M-matrix

$\Rightarrow \quad D$ CI

D Clayton or Gumbel copula $\Rightarrow$ completely monotone generator

$\Rightarrow \quad D$ CI

Example (cont.)

general dependence within subgroups

a) unconstrained bounds
$d = 50$

	$(\underline{\mathrm{VaR}}_\alpha; \overline{\mathrm{VaR}}_\alpha)$	$(a; b)$
$\alpha = 0.95$	(18; 153)	(18; 154)
$\alpha = 0.99$	(22; 298)	(22; 298)
$\alpha = 0.995$	(23; 388)	(22; 389)

b) $C \leq_{\mathrm{wcs}} D = C^\perp$ independent subgroups

	$k = 2$	$k = 5$	$k = 10$	$k = 25$
$\alpha = 0.95$	(20; 116)	(22; 82)	(23; 64)	(24; 49)
$\alpha = 0.99$	(23; 209)	(24; 132)	(24; 96)	(25; 66)
$\alpha = 0.995$	(24; 266)	(24; 163)	(25; 115)	(25; 76)

Example (cont.)

c) D Gauss copula resp. t-copula

			$k=2$	$k=5$	$k=10$	$k=25$	$\overline{\Delta}$
Tab. A:	Corr $=0.1$	$\alpha=0.95$	(20; 119)	(22; 88)	(22; 73)	(23; 71)	58
		$\alpha=0.99$	(23; 214)	(24; 142)	(24; 116)	(24; 110)	130
		$\alpha=0.995$	(24; 271)	(24; 174)	(24; 135)	(24; 131)	174
Tab. B:	Corr $=0.25$	$\alpha=0.95$	(20; 124)	(21; 98)	(22; 86)	(22; 78)	58
		$\alpha=0.99$	(23; 222)	(24; 161)	(24; 134)	(24; 115)	107
		$\alpha=0.995$	(24; 283)	(24; 197)	(24; 160)	(25; 135)	135
Tab. C:	Corr $=0.5$	$\alpha=0.95$	(19; 132)	(20; 116)	(21; 109)	(21; 105)	27
		$\alpha=0.99$	(23; 242)	(24; 200)	(23; 183)	(24; 172)	70
		$\alpha=0.995$	(24; 308)	(24; 248)	(24; 225)	(25; 210)	98
Tab. D:	$\nu=50$,	$\alpha=0.95$	(20; 119)	(22; 89)	(22; 74)	(23; 63)	56
	Corr $=0.1$	$\alpha=0.99$	(23; 215)	(24; 146)	(24; 114)	(24; 90)	125
		$\alpha=0.995$	(24; 274)	(24; 179)	(24; 137)	(25; 105)	169
Tab. E:	$\nu=50$,	$\alpha=0.95$	(20; 124)	(21; 99)	(22; 88)	(23; 80)	44
	Corr $=0.25$	$\alpha=0.99$	(23; 224)	(24; 164)	(24; 139)	(24; 122)	102
		$\alpha=0.995$	(24; 285)	(24; 202)	(24; 168)	(24; 144)	143
Tab. F:	$\nu=10$,	$\alpha=0.95$	(20; 125)	(21; 102)	(21; 93)	(23; 87)	38
	Corr $=0.25$	$\alpha=0.99$	(23; 230)	(23; 177)	(24; 157)	(24; 144)	86
		$\alpha=0.995$	(24; 294)	(24; 223)	(24; 196)	(24; 177)	117

VaR bounds in subgroup model with Gauss copula in A, B, and C and with t-copula in D, E, and F. $\overline{\Delta}$ denotes the difference between upper bounds for $k=2$ and $k=25$.

Example (cont.)

d) *D* Clayton resp. Gumble copula

			$k=2$	$k=5$	$k=10$	$k=25$	$\overline{\Delta}$
Tab. A:	$\vartheta=1$	$\alpha=0.95$	(20; 122)	(22; 94)	(22; 81)	(23; 71)	51
		$\alpha=0.99$	(23; 216)	(24; 147)	(24; 116)	(24; 92)	124
		$\alpha=0.995$	(24; 274)	(24; 179)	(24; 135)	(25; 103)	171
Tab. B:	$\vartheta=3$	$\alpha=0.95$	(20; 130)	(21; 108)	(21; 98)	(22; 90)	40
		$\alpha=0.99$	(23; 227)	(24; 166)	(24; 138)	(24; 119)	108
		$\alpha=0.995$	(24; 285)	(24; 198)	(24; 160)	(25; 132)	153
Tab C:	$\vartheta=10$	$\alpha=0.95$	(19; 140)	(20; 128)	(20; 122)	(20; 118)	22
		$\alpha=0.99$	(23; 244)	(23; 196)	(23; 176)	(24; 162)	82
		$\alpha=0.995$	(24; 304)	(24; 232)	(24; 202)	(24; 180)	124
Tab. D:	$\vartheta=1.5$	$\alpha=0.95$	(19; 140)	(19; 132)	(20; 129)	(20; 127)	13
		$\alpha=0.99$	(23; 272)	(23; 258)	(23; 254)	(23; 250)	22
		$\alpha=0.995$	(23; 353)	(23; 338)	(23; 329)	(23; 327)	26
Tab. E:	$\vartheta=3$	$\alpha=0.95$	(18; 151)	(18; 150)	(18; 149)	(18; 148)	3
		$\alpha=0.99$	(22; 294)	(22; 290)	(22; 290)	(22; 289)	5
		$\alpha=0.995$	(23; 383)	(23; 379)	(23; 379)	(23; 375)	8

VaR bounds in subgroup model with Clayton copula in A, B, and C and Gumbel copula in D and E.

7. Conclusion

- Risk bounds with marginal information can be calculated, typically (too) wide
- Various reductions by including additional information
- Higher dimensional marginals (reduced bounds)
- Variance constraints, higher order moment constraints good reduction, when constraints are small enough
- partial independence structure (combined with variance information)
  - strong reduction of dependence uncertainty
  - leads in examples to realistic bounds

Conclusion (cont.)

- reduction by partially specified risk factor models
  - strong reduction of dependence uncertainty
  - applicable in variety of mixture models
- partial dependence information (improved standard bounds)
  - one-sided bounds, improved Hoeffding–Fréchet bounds
    needs strong enough dependence constraints, d small
  - two-sided dependence bounds
    promising tool also in higher dimension d
  - subgroup structure models
    flexible tool, good reduction

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