# Robust risk aggregation with neural networks

Michael Kupper

Universität Konstanz

joint work with Stephan Eckstein and Mathias Pohl

Model Uncertainty in Risk Management
Paris, January 2020

# Motivating Example

- What is the Average-Value-at-Risk $\mathrm{AVaR}_{0.95}(U+V)$ of standard uniforms $U, V \sim \mathrm{Uni}([0,1])$?

# Motivating Example

- What is the Average-Value-at-Risk $\mathrm{AVaR}_{0.95}(U+V)$ of standard uniforms $U, V \sim \mathrm{Uni}([0,1])$?
  - Assume $U$ and $V$ are independent and hence coupled with the product copula $\Pi(u,v) = uv$ for all $u, v \in [0,1]$. Then
  $$\mathrm{AVaR}_\alpha^\Pi(U+V) = 1.789.$$
  - Assume $U$ and $V$ are comonotonic and hence coupled with the copula $M(u,v) = \min(u,v)$ for all $u, v \in [0,1]$. Then
  $$\mathrm{AVaR}_\alpha^M(U+V) = 1.95.$$
  - Assume $U$ and $V$ are counter-monotonic and hence coupled with the copula $W(u,v) = \max(u+v-1, 0)$ for all $u, v \in [0,1]$. Then
  $$\mathrm{AVaR}_\alpha^W(U+V) = 1.$$

# Motivating Example

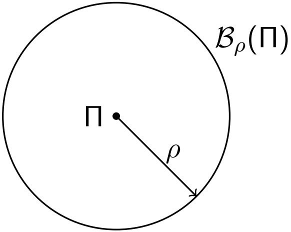
- What is the Average-Value-at-Risk $\mathrm{AVaR}_{0.95}(U+V)$ of standard uniforms $U, V \sim \mathrm{Uni}([0,1])$?
- We can only derive bounds: $1 \leq \mathrm{AVaR}_{0.95}(U+V) \leq 1.95$

# Motivating Example

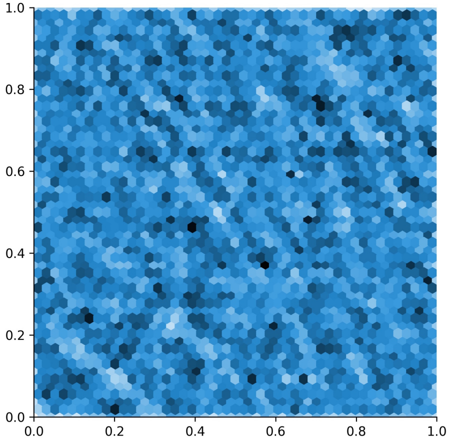
- What is the Average-Value-at-Risk $\mathrm{AVaR}_{0.95}(U+V)$ of standard uniforms $U, V \sim \mathrm{Uni}([0,1])$?
- We can only derive bounds: $1 \leq \mathrm{AVaR}_{0.95}(U+V) \leq 1.95$
- How can we incorporate the believe that $U$ and $V$ are independent to derive tighter bounds?

# Motivating Example

- What is the Average-Value-at-Risk $\mathrm{AVaR}_{0.95}(U+V)$ of standard uniforms $U, V \sim \mathrm{Uni}([0,1])$?
- We can only derive bounds: $1 \leq \mathrm{AVaR}_{0.95}(U+V) \leq 1.95$
- How can we incorporate the believe that $U$ and $V$ are independent to derive tighter bounds?
  - We account for model/dependence unvertainty with respect to the product copula $\Pi$.
  - We can consider an *appropriate neighborhood* $\mathcal{B}_\rho(\Pi)$ of the reference dependence structure $\Pi$, rather than all possible dependence structures.

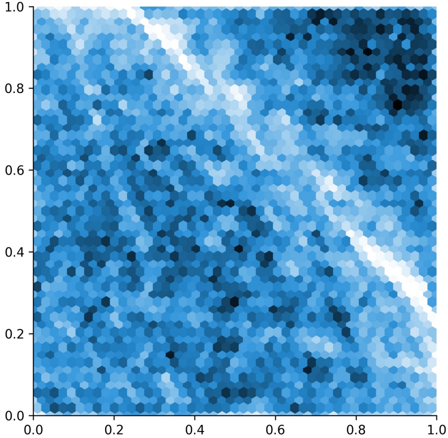


# Motivating Example



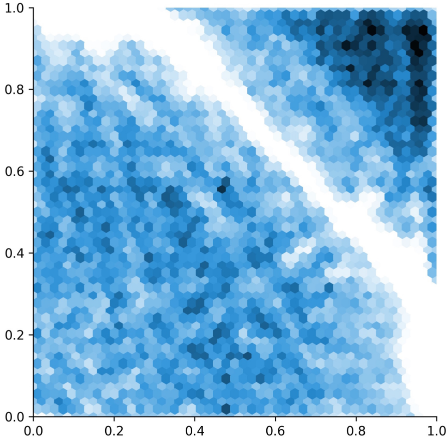

Samples from the optimizer $C$ of $\sup_{\binom{V}{U}\sim C\in\mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U+V)$ for $\rho = 0$.

# Motivating Example



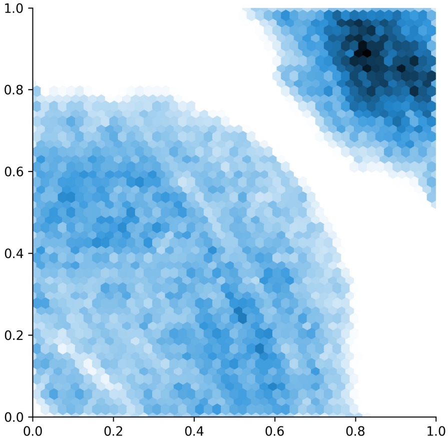

Samples from the optimizer $C$ of $\sup_{\binom{V}{U} \sim C \in \mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U + V)$ for $\rho = 0.04$.

# Motivating Example



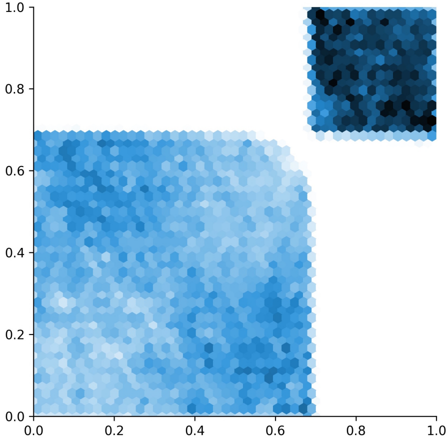

Samples from the optimizer $C$ of $\sup_{\binom{V}{U} \sim C \in \mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U+V)$ for $\rho = 0.08$.

# Motivating Example



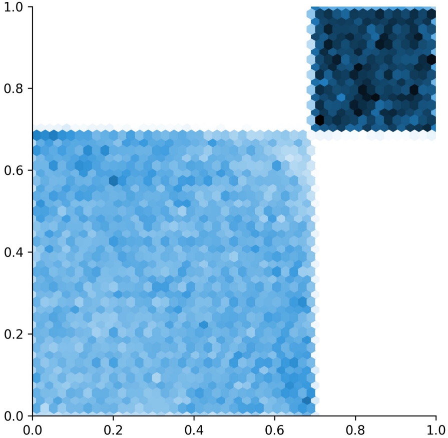

Samples from the optimizer $C$ of $\sup_{\binom{V}{U}\sim C\in\mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U+V)$ for $\rho = 0.12$.

# Motivating Example




Samples from the optimizer $C$ of $\sup_{\binom{V}{U}\sim C\in\mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U+V)$ for $\rho = 0.16$.

# Motivating Example




Samples from the optimizer $C$ of $\sup_{\binom{V}{U}\sim C\in\mathcal{B}_\rho(\Pi)} \mathrm{AVaR}_\alpha(U+V)$ for $\rho = 0.20$.

# Outline

1. Risk aggregation

2. Penalization of superhedging problems

3. Examples

# Robust risk aggregation

Computation of the worst case average value at risk of a sum of dependent random variables:

$$\sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\leq\rho}} AVaR_\alpha^P(X_1+\dots+X_d)$$

$$= \sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\leq\rho}} \inf_{\lambda\in\mathbb{R}} \left\{ E^P\left[\frac{(X_1+\dots+X_d-\lambda)^+}{\alpha}\right]+\lambda\right\}$$

$$= \inf_{\lambda\in\mathbb{R}} \sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\leq\rho}} \left\{ E^P\left[\frac{(X_1+\dots+X_d-\lambda)^+}{\alpha}\right]+\lambda\right\}$$

# Robust risk aggregation

Computation of the worst case average value at risk of a sum of dependent random variables:

$$\sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\le\rho}} AVaR_\alpha^P(X_1+\cdots+X_d)$$

$$= \sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\le\rho}} \inf_{\lambda\in\mathbb{R}} \left\{ E^P\left[\frac{(X_1+\cdots+X_d-\lambda)^+}{\alpha}\right]+\lambda\right\}$$

$$= \inf_{\lambda\in\mathbb{R}} \sup_{\substack{P\circ X_i^{-1}\sim\bar{\mu}_i \\ d_c(P\circ X^{-1},\bar{\mu})\le\rho}} \left\{ E^P\left[\frac{(X_1+\cdots+X_d-\lambda)^+}{\alpha}\right]+\lambda\right\}$$

# Expectations under dependence uncertainty

Our goal is to compute

$$\max_{\substack{\mu \in \Pi(\bar{\mu}_1, \ldots, \bar{\mu}_d) \\ d_c(\mu, \bar{\mu}) \leq \rho}} \int_{\mathbb{R}^d} f \, d\mu$$

where $\bar{\mu}$ is a reference probability measure on $\mathbb{R}^d$,
$\Pi(\bar{\mu}_1, \ldots, \bar{\mu}_d)$ denotes the set of all couplings with marginals $\bar{\mu}_1, \ldots, \bar{\mu}_d$.
Here, we consider a transport distance

$$d_c(\mu, \bar{\mu}) := \inf_{\pi \in \Pi(\bar{\mu}, \mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\,(x, y)\, \pi(dx, dy)$$

e.g. $c(x, y) = \sum_i |x_i - y_i|$.

# Expectations under dependence uncertainty

**Theorem**

*For every $f \in U_b(\mathbb{R}^d)$ it holds*

$$\max_{\substack{\mu \in \Pi(\bar\mu_1,\ldots,\bar\mu_d) \\ d_c(\mu,\bar\mu)\le\rho}} \int_{\mathbb{R}^d} f \, d\mu$$

$$= \inf_{\lambda\ge 0,\, h_i \in C_b(\mathbb{R})} \left\{ \rho\lambda + \sum_{i=1}^d \int_{\mathbb{R}} h_i \, d\bar\mu_i + \int_{\mathbb{R}^d} \sup_{y\in\mathbb{R}^d} \left[ f(y) - \sum_{i=1}^d h_i(y_i) - \lambda c(x,y) \right] \bar\mu(dx) \right\}$$

*for each radius $\rho \ge 0$ and a every reference measure $\bar\mu \in \Pi(\bar\mu_1, \ldots, \bar\mu_d)$.*

See also Esfahani and Kuhn (2016), Blanchet and Murthy (2016), Gao and Kleywegt (2017), Bartl, Drapeau, Tangpi (2017)

# Expectations under dependence uncertainty

## Theorem

*For every $f \in U_b(\mathbb{R}^d)$ it holds*

$$\max_{\substack{\mu \in \Pi(\bar{\mu}_1,\dots,\bar{\mu}_d) \\ d_c(\mu,\bar{\mu}) \le \rho}} \int_{\mathbb{R}^d} f \, d\mu$$

$$= \inf_{\lambda \ge 0,\, h_i \in C_b(\mathbb{R})} \left\{ \rho\lambda + \sum_{i=1}^d \int_{\mathbb{R}} h_i \, d\bar{\mu}_i + \int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} \left[ f(y) - \sum_{i=1}^d h_i(y_i) - \lambda c(x,y) \right] \bar{\mu}(dx) \right\}$$

$$= \inf_{\substack{\lambda \ge 0,\, h_i \in C_b(\mathbb{R}) \\ g \in C_b(\mathbb{R}^d) \\ g(x) \ge f(y) - \sum_{i=1}^d h_i(y_i) - \lambda c(x,y)}} \left\{ \lambda\rho + \sum_{i=1}^d \int_{\mathbb{R}} h_i \, d\bar{\mu}_i + \int_{\mathbb{R}^d} g(x) \, \bar{\mu}(dx) \right\}$$

*for each radius $\rho \ge 0$ and a every reference measure $\bar{\mu} \in \Pi(\bar{\mu}_1, \dots, \bar{\mu}_d)$.*

# Penalization of superhedging problems

# Robust optimization problem

**Objective:** Solve

$$\sup_{\nu \in \mathcal{Q}} \int f \, d\nu$$

where

- $\mathcal{Q}$ is a set of probability measures on $\mathbb{R}^d$
- $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded

# Robust optimization problem

**Objective:** Solve

$$\sup_{\nu \in \mathcal{Q}} \int f \, d\nu = \inf_{\substack{h \in \mathcal{H} \\ h \geq f}} \int h \, d\mu_0$$

where

- $\mathcal{Q}$ is a set of probability measures on $\mathbb{R}^d$
- $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded
- $\mathcal{H} \subseteq C_b(\mathbb{R}^d)$
- $\mu_0$ is a probability measure on $\mathbb{R}^d$

# Penalization

| superhedging problem | penalized version |
| --- | --- |
| $(D) = \inf_{\substack{h \in \mathcal{H} \\ h \geq f}} \int h \, d\mu_0$ | $(D_{\theta,\gamma}) = \inf_{h \in \mathcal{H}} \int h \, d\mu_0 + \int \beta_\gamma(f - h) d\theta$ |
| | |

where

- $\beta_\gamma$ is a differentiable nondecreasing convex function, parameterized by $\gamma \in \mathbb{R}_+$ (e.g. $\beta_\gamma = \gamma \max\{0, x\}^2$)
- $\theta$ is a probability measure on $\mathbb{R}^d$

# Penalization

| superhedging problem | penalized version |
|---|---|
| $(D) = \inf_{\substack{h\in\mathcal{H}\\ h\geq f}} \int h\,d\mu_0$ | $(D_{\theta,\gamma}) = \inf_{h\in\mathcal{H}} \int h\,d\mu_0 + \int \beta_\gamma(f-h)d\theta$ |
| $(P) = \sup_{\nu\in\mathcal{Q}} \int f\,d\nu$ | $(P_{\theta,\gamma}) = \sup_{\nu\in\mathcal{Q}} \int fd\nu - \int \beta^*_\gamma\left(\frac{d\nu}{d\theta}\right)d\theta$ |

If $\hat{h}$ is an optimizer of $(D_{\theta,\gamma})$, then $\hat{\nu}$ given by

$$\frac{d\hat{\nu}}{d\theta} = \beta'_\gamma(f-\hat{h})$$

is an optimizer of $(P_{\theta,\gamma})$.

# Approximation with neural networks

The penalized version

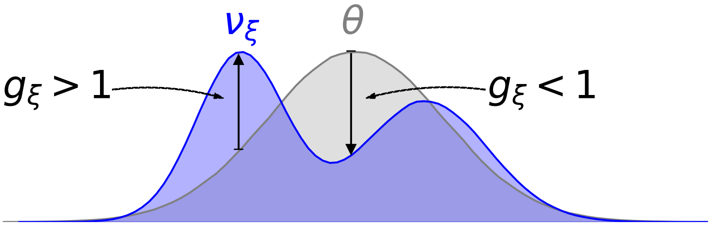
$$(D_{\theta,\gamma}) = \inf_{h \in \mathcal{H}} \int h \, d\mu_0 + \int \beta_\gamma(f - h) d\theta$$

can be solved by replacing $\mathcal{H}$ by a set of neural network functions $\mathcal{H}^m$. This leads to the finite-dimensional problem

$$(D^m_{\theta,\gamma}) = \inf_{h \in \mathcal{H}^m} \int h \, d\mu_0 + \int \beta_\gamma(f - h) d\theta$$

# Parametric representations of probability measures

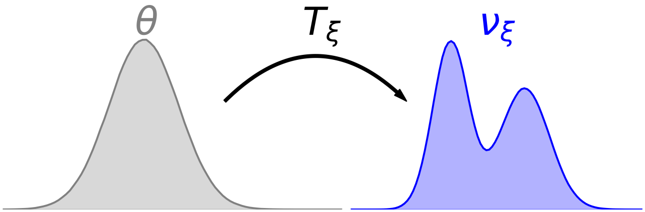
Density




$\frac{d\nu_\xi}{d\theta} = g_\xi$, where $g_\xi$ is a NN function and $\theta$ is some reference measure

# Parametric representations of probability measures

Pushforward




$\nu_\xi = \theta \circ T_\xi^{-1}$, where $T_\xi$ is a NN function and $\theta$ is some reference measure

# Pushforward representation: Min-Max formulation

- $(P) = \sup_{\nu \in \mathcal{Q}} \int f \, d\nu$
- $\mathcal{Q} = \{\nu \in \mathcal{P}(\mathbb{R}^d) : \int h \, d\nu = \int h \, d\mu$ for all $h \in \mathcal{H}\}$ where $\mathcal{H} \subseteq C(\mathbb{R}^d)$ is a linear space of functions, $\mu \in \mathcal{P}(\mathbb{R}^d)$ fix.

Let $\theta \in \mathcal{P}(\mathbb{R}^K)$ be sufficiently rich, e.g. $\theta = \mathcal{U}([0,1]^K)$. For a function $T$ denote by $\theta_T := \theta \circ T^{-1}$ the pushforward of $\theta$ under $T$. Then

# Pushforward representation: Min-Max formulation

- $(P) = \sup_{\nu \in \mathcal{Q}} \int f \, d\nu$
- $\mathcal{Q} = \{\nu \in \mathcal{P}(\mathbb{R}^d) : \int h \, d\nu = \int h \, d\mu \text{ for all } h \in \mathcal{H}\}$ where $\mathcal{H} \subseteq C(\mathbb{R}^d)$ is a linear space of functions, $\mu \in \mathcal{P}(\mathbb{R}^d)$ fix.

Let $\theta \in \mathcal{P}(\mathbb{R}^K)$ be sufficiently rich, e.g. $\theta = \mathcal{U}([0,1]^K)$. For a function $T$ denote by $\theta_T := \theta \circ T^{-1}$ the pushforward of $\theta$ under $T$. Then

$$
\begin{aligned}
(P) &= \sup_{\nu \in \mathcal{P}(\mathbb{R}^d)} \inf_{h \in \mathcal{H}} \int f \, d\nu + \int h \, d\nu - \int h \, d\mu \\
&= \sup_{T : \mathbb{R}^K \to \mathbb{R}^d} \inf_{h \in \mathcal{H}} \int f(T) \, d\theta_T - \int h \, d\theta_T - \int h \, d\mu
\end{aligned}
$$

# Pushforward representation: Min-Max formulation

- $(P) = \sup_{\nu \in \mathcal{Q}} \int f \, d\nu$
- $\mathcal{Q} = \{\nu \in \mathcal{P}(\mathbb{R}^d) : \int h \, d\nu = \int h \, d\mu \text{ for all } h \in \mathcal{H}\}$ where $\mathcal{H} \subseteq C(\mathbb{R}^d)$ is a linear space of functions, $\mu \in \mathcal{P}(\mathbb{R}^d)$ fix.

Let $\theta \in \mathcal{P}(\mathbb{R}^K)$ be sufficiently rich, e.g. $\theta = \mathcal{U}([0,1]^K)$. For a function $T$ denote by $\theta_T := \theta \circ T^{-1}$ the pushforward of $\theta$ under $T$. Then

$$
\begin{aligned}
(P) &= \sup_{\nu \in \mathcal{P}(\mathbb{R}^d)} \inf_{h \in \mathcal{H}} \int f \, d\nu + \int h \, d\nu - \int h \, d\mu \\
&= \sup_{T : \mathbb{R}^K \to \mathbb{R}^d} \inf_{h \in \mathcal{H}} \int f(T) \, d\theta_T - \int h \, d\theta_T - \int h \, d\mu
\end{aligned}
$$

Adapt methods from Generative Adversarial Models:

- Relaxations or Regularizations of the objective function
- Using game theoretic considerations, like mixing strategies or anticipating the other 'player'

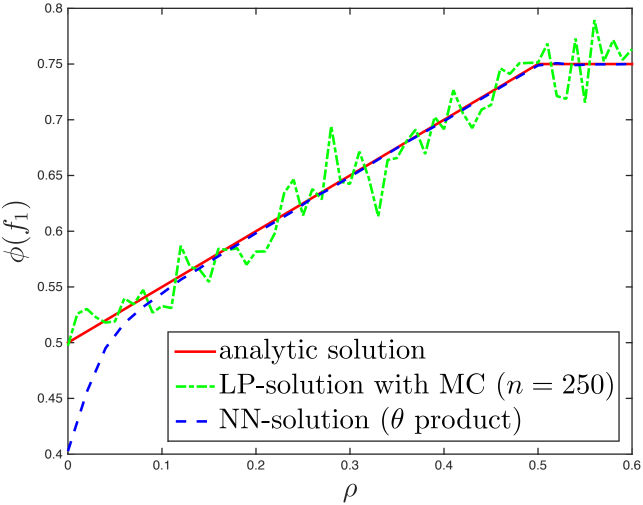
# Examples

# Expected maximum of two comonotone standard Uniforms

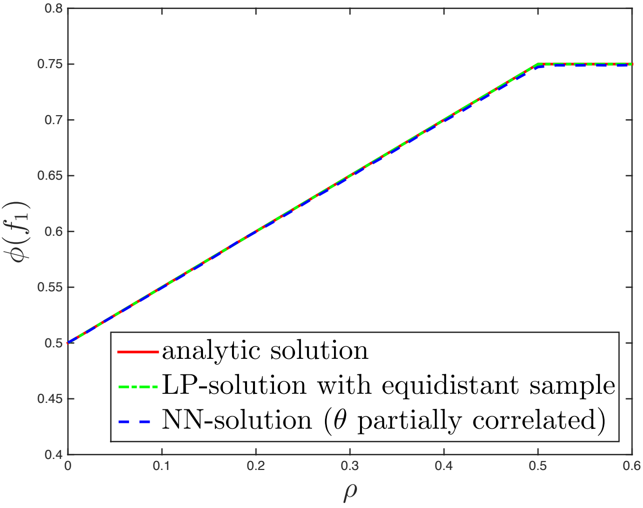
$$\phi(f_1) := \sup_{\substack{\binom{V}{U} \sim \mu \in \Pi(\bar{\mu}_1, \bar{\mu}_2), \\ d_c(\bar{\mu}, \mu) \leq \rho}} \mathbb{E}\left[\max(U, V)\right] = \frac{1 + \min(\rho, 0.5)}{2}$$

where $\bar{\mu}_1 = \bar{\mu}_2 = \mathcal{U}([0,1])$ are standard uniformly distributed probability measures, $\bar{\mu}$ is the comonotone copula and $c(x, y) = ||x - y||_1$.

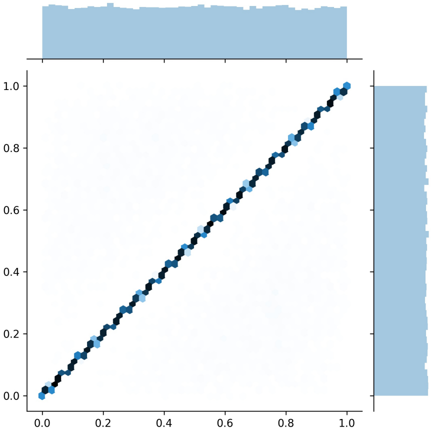
# Expected maximum of two comonotone standard Uniforms



# Expected maximum of two comonotone standard Uniforms

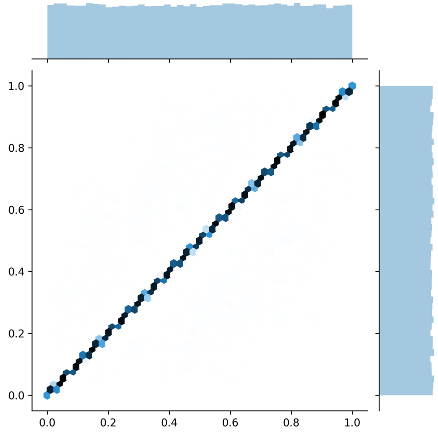


# Expected maximum of two comonotone standard Uniforms



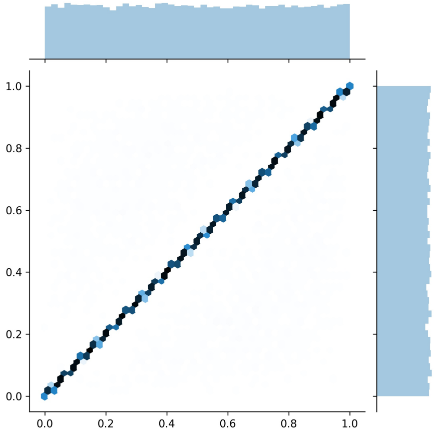

Samples from the optimizer $\mu$ for $\rho = 0$.

# Expected maximum of two comonotone standard Uniforms



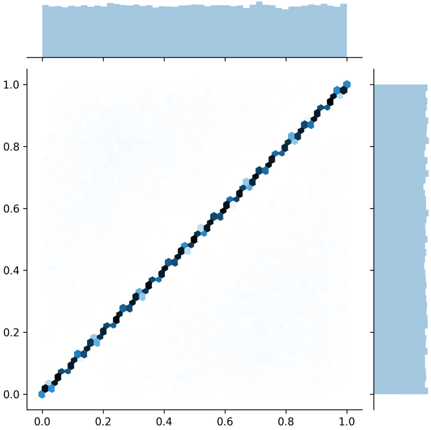

Samples from the optimizer $\mu$ for $\rho = 0.03$.

# Expected maximum of two comonotone standard Uniforms



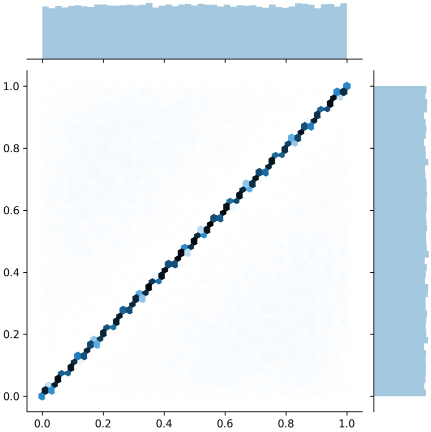

Samples from the optimizer $\mu$ for $\rho = 0.06$.

# Expected maximum of two comonotone standard Uniforms



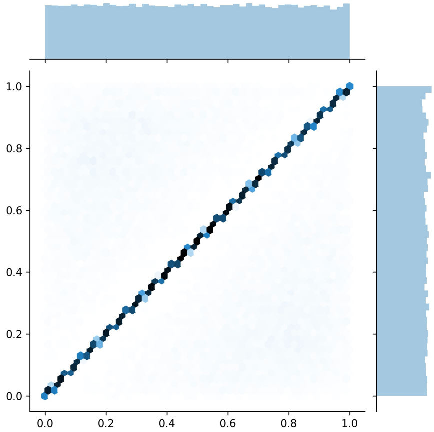

Samples from the optimizer $\mu$ for $\rho = 0.1$.

# Expected maximum of two comonotone standard Uniforms



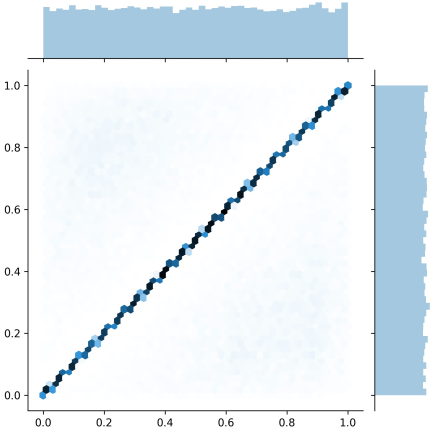

Samples from the optimizer $\mu$ for $\rho = 0.13$.

# Expected maximum of two comonotone standard Uniforms



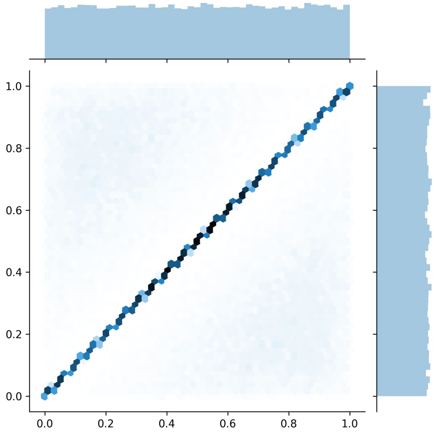

Samples from the optimizer $\mu$ for $\rho = 0.16$.

# Expected maximum of two comonotone standard Uniforms



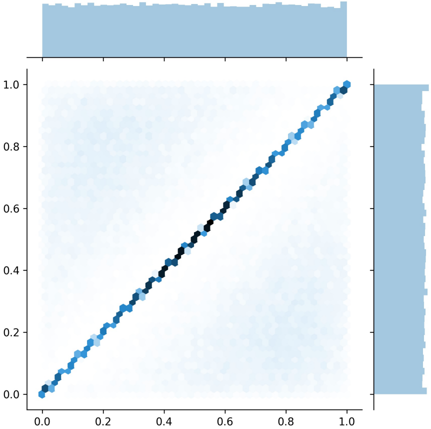

Samples from the optimizer $\mu$ for $\rho = 0.19$.

# Expected maximum of two comonotone standard Uniforms



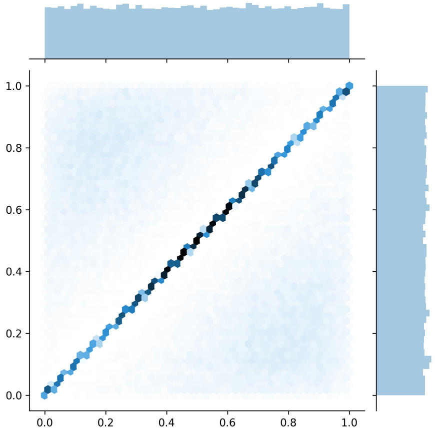

Samples from the optimizer $\mu$ for $\rho = 0.23$.

# Expected maximum of two comonotone standard Uniforms



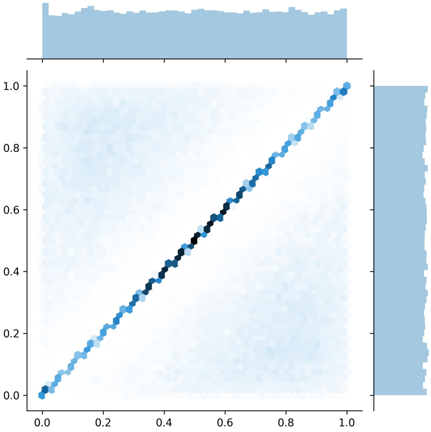

Samples from the optimizer $\mu$ for $\rho = 0.26$.

# Expected maximum of two comonotone standard Uniforms



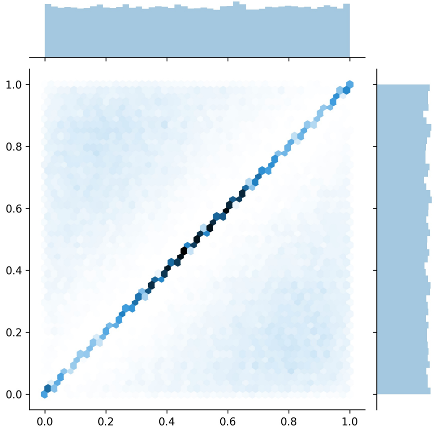

Samples from the optimizer $\mu$ for $\rho = 0.29$.

# Expected maximum of two comonotone standard Uniforms



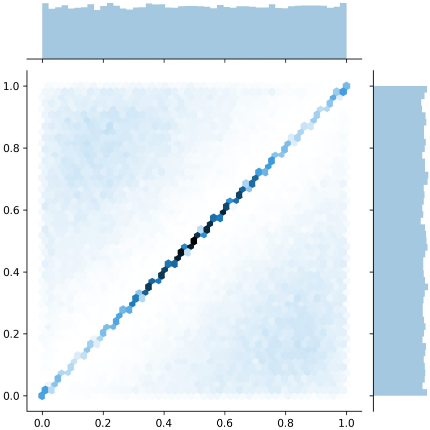

Samples from the optimizer $\mu$ for $\rho = 0.32$.

# Expected maximum of two comonotone standard Uniforms



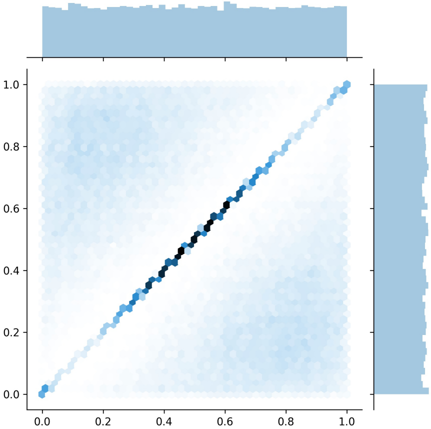

Samples from the optimizer $\mu$ for $\rho = 0.36$.

# Expected maximum of two comonotone standard Uniforms



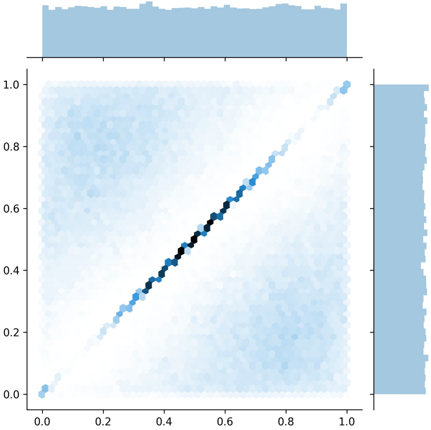

Samples from the optimizer $\mu$ for $\rho = 0.39$.

# Expected maximum of two comonotone standard Uniforms



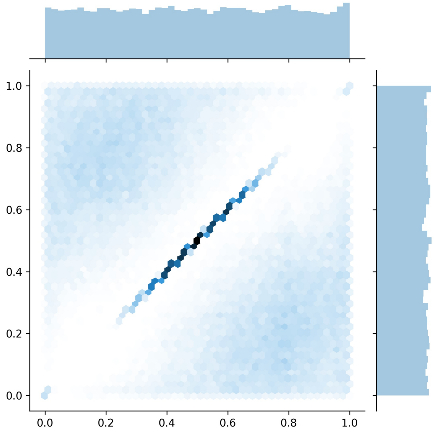

Samples from the optimizer $\mu$ for $\rho = 0.4$.

# Expected maximum of two comonotone standard Uniforms



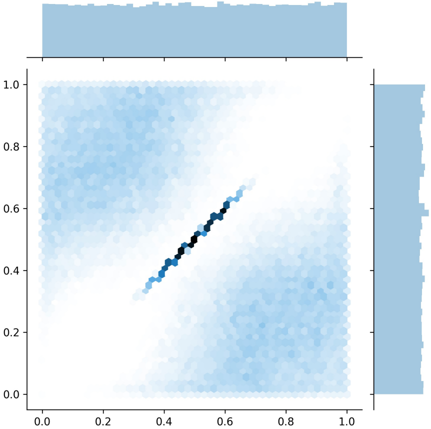

Samples from the optimizer $\mu$ for $\rho = 0.42$.

# Expected maximum of two comonotone standard Uniforms



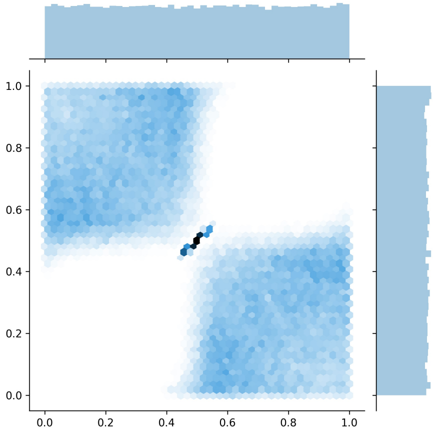

Samples from the optimizer $\mu$ for $\rho = 0.45$.

# Expected maximum of two comonotone standard Uniforms



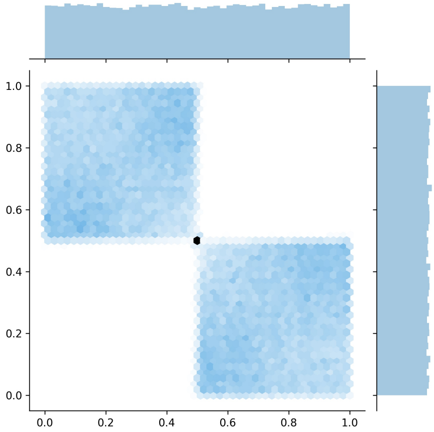

Samples from the optimizer $\mu$ for $\rho = 0.48$.

# Expected maximum of two comonotone standard Uniforms



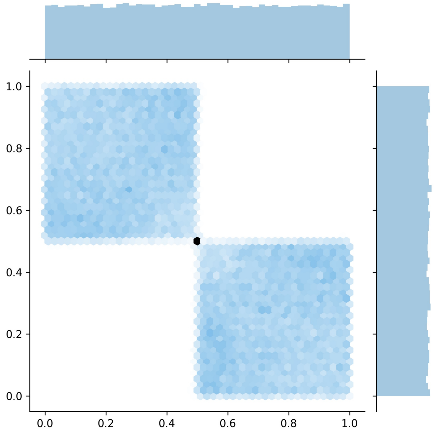

Samples from the optimizer $\mu$ for $\rho = 0.50$.

# Expected maximum of two comonotone standard Uniforms




Samples from the optimizer $\mu$ for $\rho = 0.52$.

# Expected maximum of two comonotone standard Uniforms




Samples from the optimizer $\mu$ for $\rho = 0.55$.

# Expected maximum of two comonotone standard Uniforms

Consider two different transport distances:

$$\phi(f_1) := \sup_{\binom{V}{U} \sim \mu \in \Pi(\bar{\mu}_1, \bar{\mu}_2),\, d_c(\bar{\mu}, \mu) \leq \rho} \mathbb{E}\left[\max(U, V)\right]$$

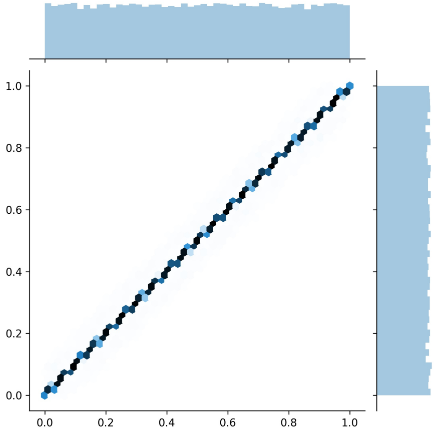
and

$$\tilde{\phi}(f_1) := \sup_{\binom{V}{U} \sim \mu \in \Pi(\bar{\mu}_1, \bar{\mu}_2),\, d_{\tilde{c}}(\bar{\mu}, \mu)^{1/2} \leq \rho} \mathbb{E}\left[\max(U, V)\right]$$

where

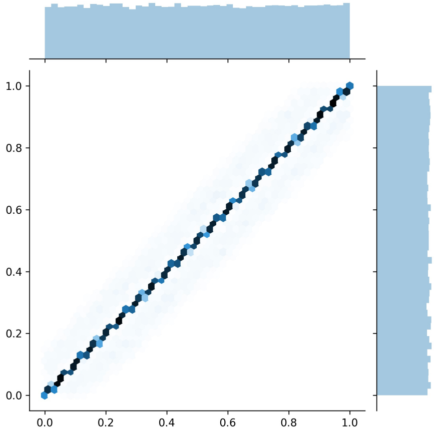
$$c(x, y) = ||x - y||_1 \quad \text{and} \quad \tilde{c}(x, y) = ||x - y||_2^2.$$

# Expected maximum of two comonotone standard Uniforms



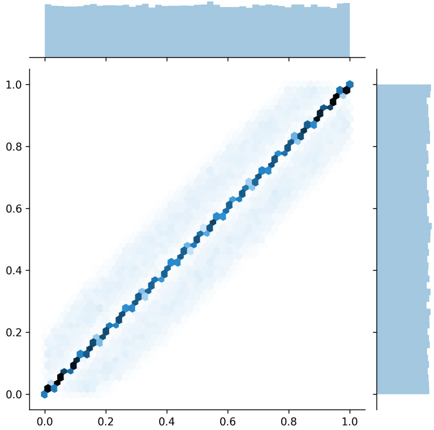

Samples from the optimizer $\mu$ for $\rho = 0$.

# Expected maximum of two comonotone standard Uniforms



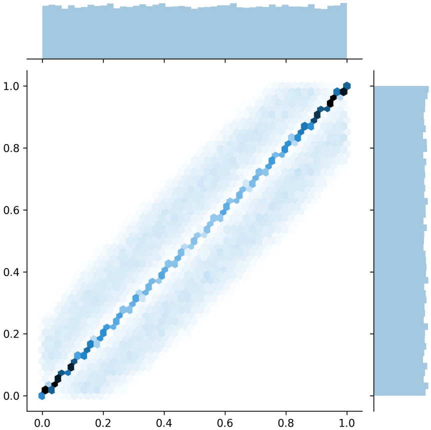

Samples from the optimizer $\mu$ for $\rho = 0.03$.

# Expected maximum of two comonotone standard Uniforms



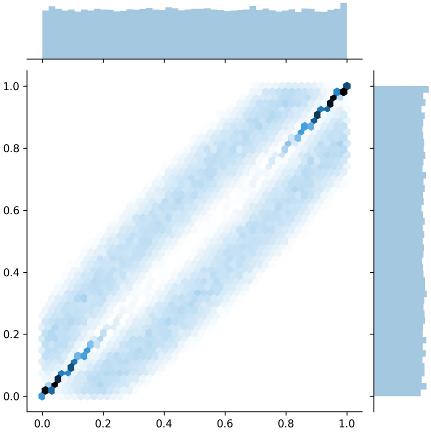

Samples from the optimizer $\mu$ for $\rho = 0.06$.

# Expected maximum of two comonotone standard Uniforms



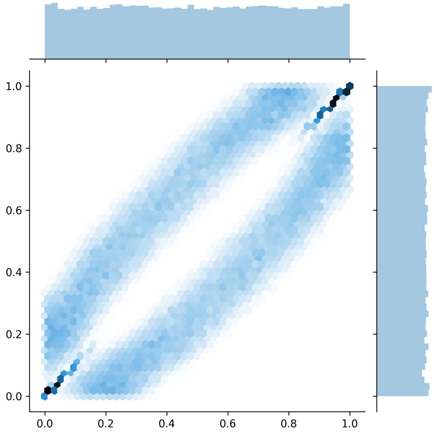

Samples from the optimizer $\mu$ for $\rho = 0.1$.

# Expected maximum of two comonotone standard Uniforms



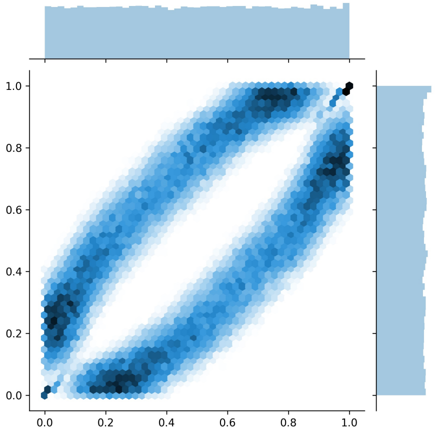

Samples from the optimizer $\mu$ for $\rho = 0.13$.

# Expected maximum of two comonotone standard Uniforms



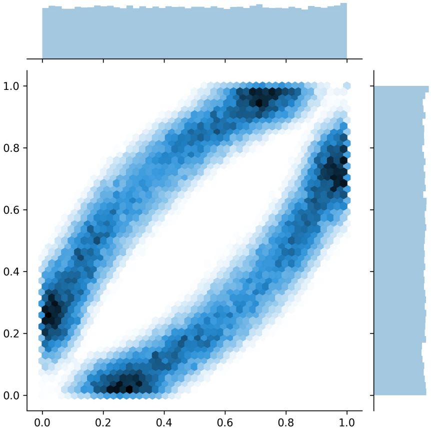

Samples from the optimizer $\mu$ for $\rho = 0.16$.

# Expected maximum of two comonotone standard Uniforms



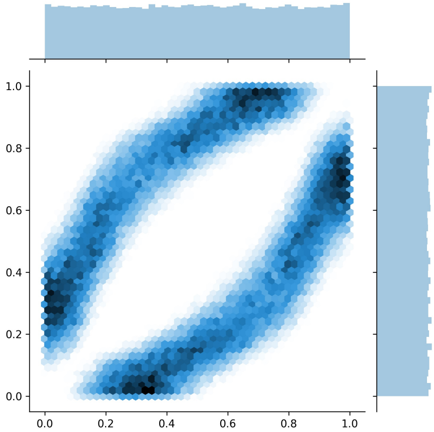

Samples from the optimizer $\mu$ for $\rho = 0.19$.

# Expected maximum of two comonotone standard Uniforms



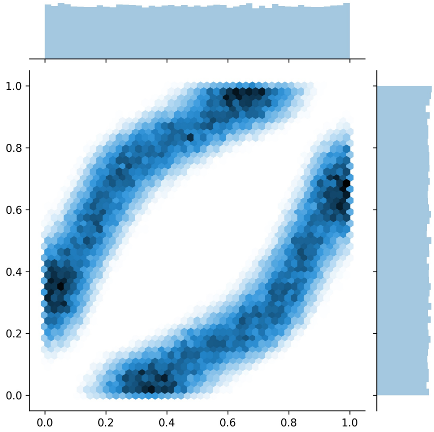

Samples from the optimizer $\mu$ for $\rho = 0.23$.

# Expected maximum of two comonotone standard Uniforms



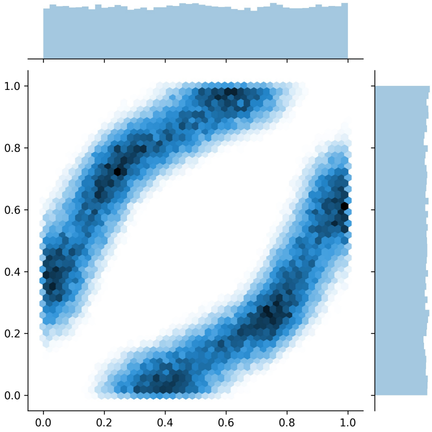

Samples from the optimizer $\mu$ for $\rho = 0.26$.

# Expected maximum of two comonotone standard Uniforms



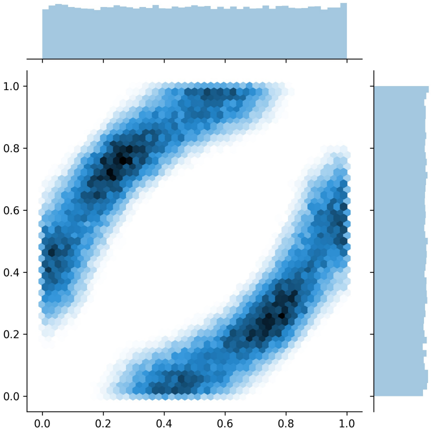

Samples from the optimizer $\mu$ for $\rho = 0.29$.

# Expected maximum of two comonotone standard Uniforms



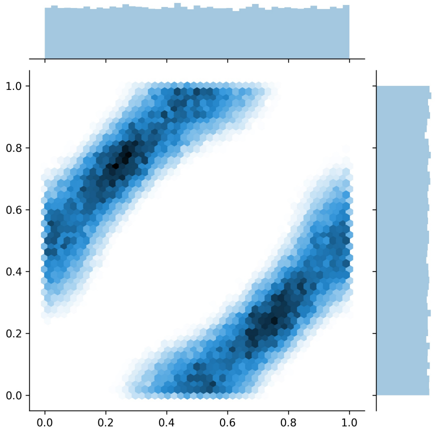

Samples from the optimizer $\mu$ for $\rho = 0.32$.

# Expected maximum of two comonotone standard Uniforms



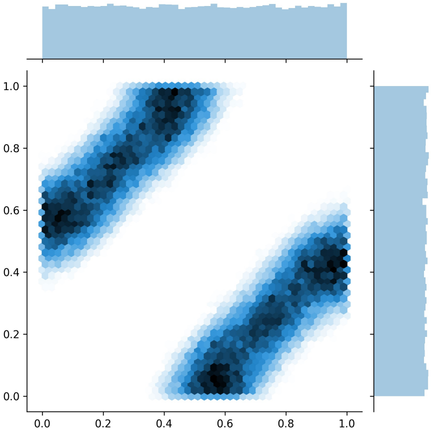

Samples from the optimizer $\mu$ for $\rho = 0.36$.

# Expected maximum of two comonotone standard Uniforms



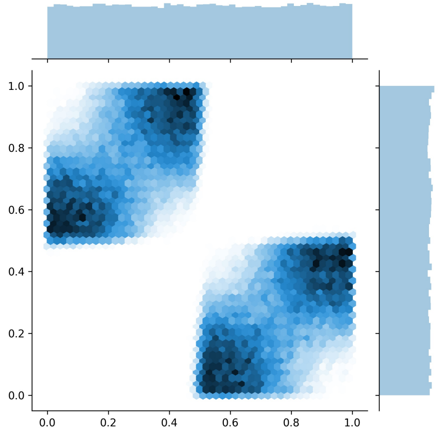

Samples from the optimizer $\mu$ for $\rho = 0.39$.

# Expected maximum of two comonotone standard Uniforms



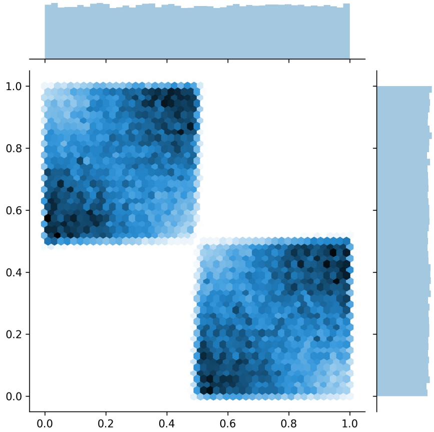

Samples from the optimizer $\mu$ for $\rho = 0.42$.

# Expected maximum of two comonotone standard Uniforms



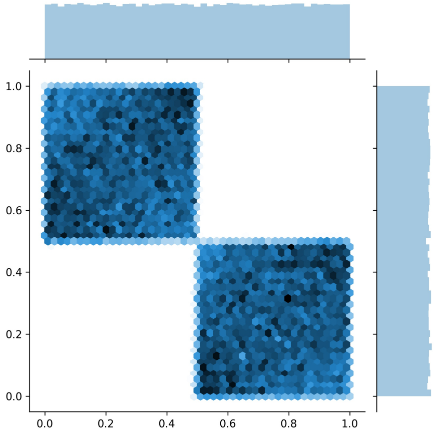

Samples from the optimizer $\mu$ for $\rho = 0.45$.

# Expected maximum of two comonotone standard Uniforms



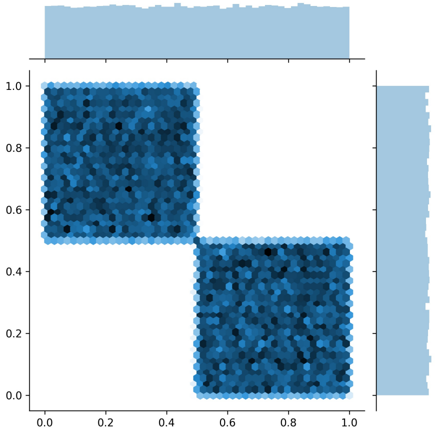

Samples from the optimizer $\mu$ for $\rho = 0.48$.

# Expected maximum of two comonotone standard Uniforms




Samples from the optimizer $\mu$ for $\rho = 0.52$.

# Expected maximum of two comonotone standard Uniforms




Samples from the optimizer $\mu$ for $\rho = 0.55$.

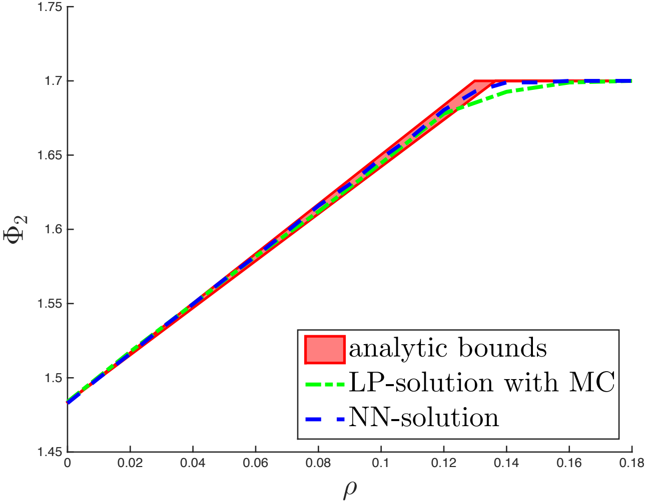
# Average Value at Risk of two independent standard Uniforms

$$\Phi_2 := \sup_{\substack{\binom{V}{U}\sim\mu\in\Pi(\bar\mu_1,\bar\mu_2),\\ d_c(\bar\mu,\mu)\le\rho}} \mathrm{AVaR}_\alpha(U+V)$$

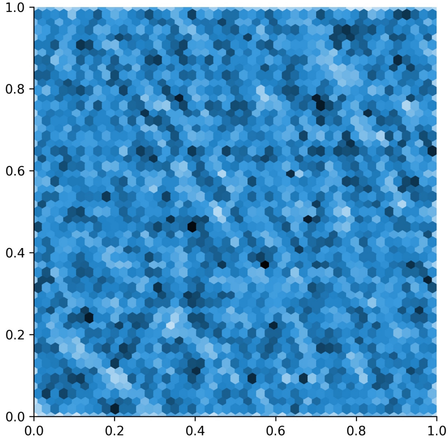
$$= \sup_{\substack{\mu\in\Pi(\bar\mu_1,\bar\mu_2),\\ d_c(\bar\mu,\mu)\le\rho}} \inf_{\tau\in\mathbb{R}} \left\{ \tau + \frac{1}{1-\alpha}\int_{[0,1]^2} \max(x_1+x_2-\tau,0)\mu(dx) \right\}$$

where $\bar\mu = \mathcal{U}([0,1]^2)$, $\bar\mu_1 = \bar\mu_2 = \mathcal{U}([0,1])$ and $c(x,y) = ||x-y||_1$.

# Average Value at Risk of two independent standard Uniforms

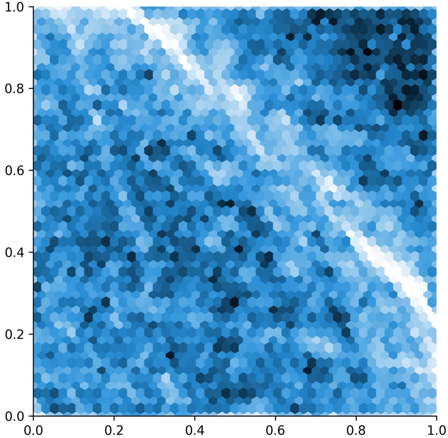


# Average Value at Risk of two independent standard Uniforms



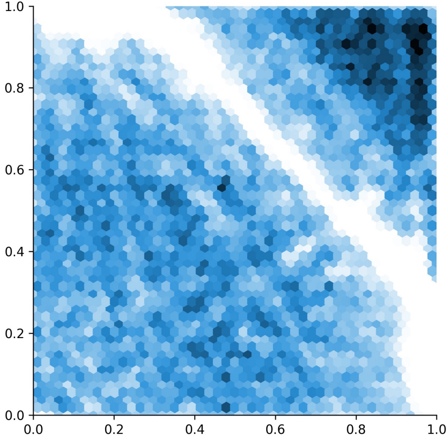

Samples from the optimizer $\mu$ for $\rho = 0$.

# Average Value at Risk of two independent standard Uniforms



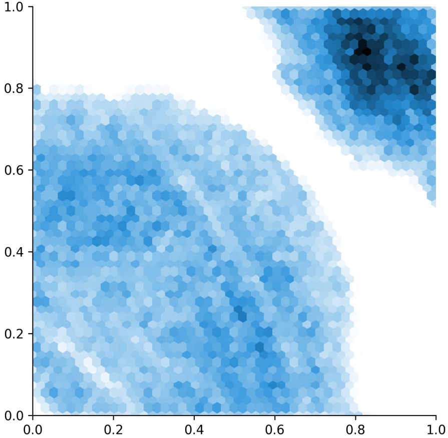

Samples from the optimizer $\mu$ for $\rho = 0.04$.

# Average Value at Risk of two independent standard Uniforms



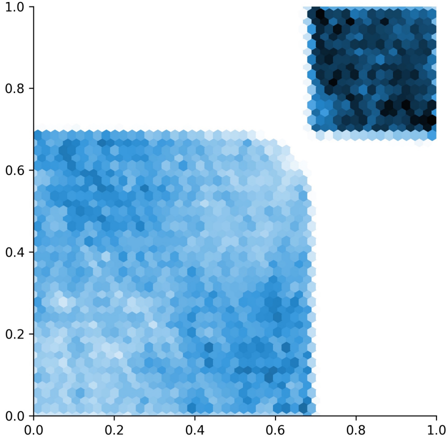

Samples from the optimizer $\mu$ for $\rho = 0.08$.

# Average Value at Risk of two independent standard Uniforms



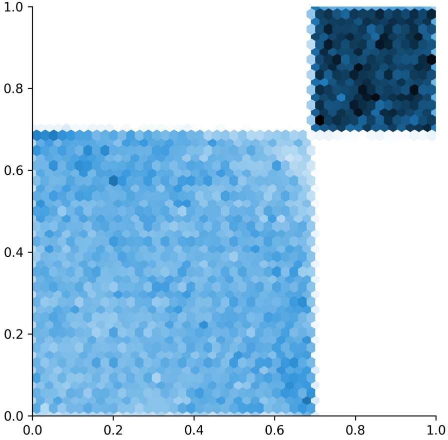

Samples from the optimizer $\mu$ for $\rho = 0.12$.

# Average Value at Risk of two independent standard Uniforms



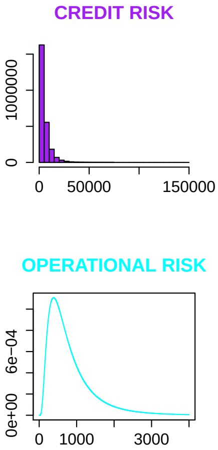

Samples from the optimizer $\mu$ for $\rho = 0.16$.

# Average Value at Risk of two independent standard Uniforms




Samples from the optimizer $\mu$ for $\rho = 0.20$.

# DNB case study



# DNB case study

| | Description | Type | Parameters/Other details |
|---|---|---|---|
| $F_1$ | cdf of credit risk $L_1$ | empirical cdf | given by 2.5 Million samples; standard deviation $\bar{\sigma}_1 = 644.602$ |
| $F_2$ | cdf of market risk $L_2$ | empirical cdf | given by 2.5 Million samples; standard deviation $\bar{\sigma}_2 = 5562.362$ |
| $F_3$ | cdf of asset risk $L_3$ | empirical cdf | given by 2.5 Million samples; standard deviation $\bar{\sigma}_3 = 1112.402$ |
| $F_4$ | cdf of operational risk $L_4$ | lognormal cdf | $\mu = 6.4741049$ and $\varsigma = 0.7213475$; standard deviation $\bar{\sigma}_4 = 694.613$ |
| $F_5$ | cdf of business risk $L_5$ | lognormal cdf | $\mu = 6.445997$ and $\varsigma = 0.574740$; standard deviation $\bar{\sigma}_5 = 465.064$ |
| $F_6$ | cdf of insurance risk $L_6$ | lognormal cdf | $\mu = 6.0534537$ and $\varsigma = 0.2489763$; standard deviation $\bar{\sigma}_6 = 111.011$ |
| $C_0$ | reference copula linking $L_1, \ldots, L_6$ | student-t copula | with 6 degrees of freedom and correlation matrix $\Sigma_0$ |

# DNB case study

We aim to compute

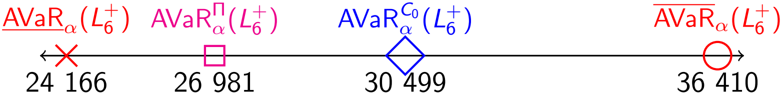
$$\underline{\Phi}_4^{C_0}(\alpha,\rho) := \inf_{\substack{L_6^+\sim\mu\in\Pi(\bar\mu_1,\ldots,\bar\mu_6),\\ d_c(\bar\mu,\mu)\le\rho}} \mathrm{AVaR}_\alpha\left(L_6^+\right),$$

$$\overline{\Phi}_4^{C_0}(\alpha,\rho) := \sup_{\substack{L_6^+\sim\mu\in\Pi(\bar\mu_1,\ldots,\bar\mu_6),\\ d_c(\bar\mu,\mu)\le\rho}} \mathrm{AVaR}_\alpha\left(L_6^+\right),$$

where $L_6^+ := \sum_{i=1}^6 L_i$ and

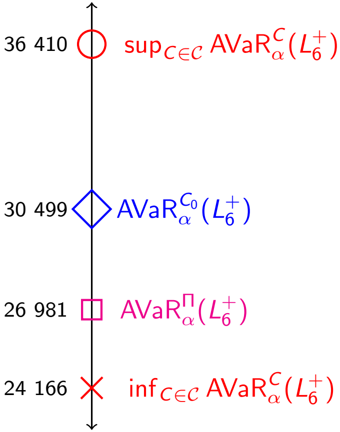
$$c(x,y) = \sum_{i=1}^{6} \frac{|x_i - y_i|}{\bar\sigma_i}.$$

# Motivating Example

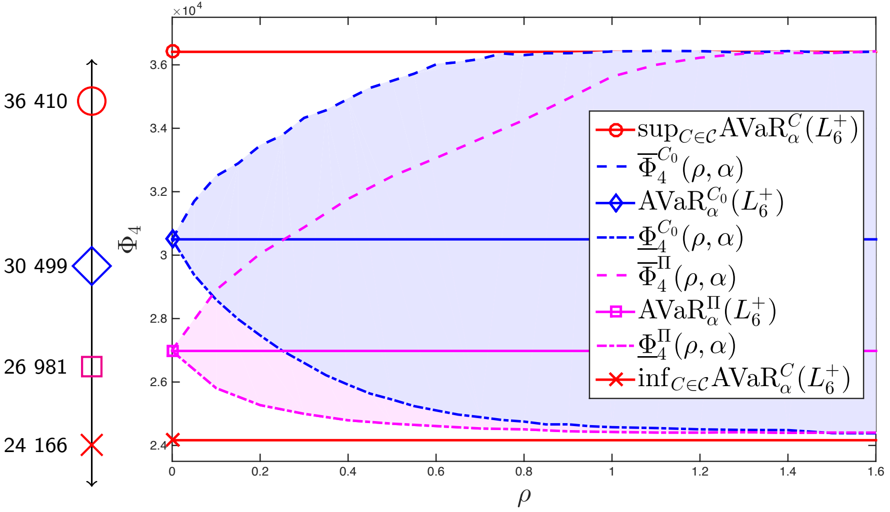



AVaR bounds for the example considered by Aas and Puccetti (2014) for $\alpha = 0.95$.

# Motivating Example



# Motivating Example



# Thank you

# References


Aas, K., and Puccetti G. (2014)
*Bounds on total economic capital: the DNB case study*
Extremes 17.4 (2014): 693-715.

Gao R. and Kleywegt A.J. (2017)
*Data-driven robust optimization with known marginal distribution*
https://faculty.mccombs.utexas.edu/rui.gao/copula.pdf

Bartl, D. and Drapeau, S. and Tangpi, L. (2018)
*Computational aspects of robust optimized certainty equivalents and option pricing*
Mathematical Finance (2019): 1-23. https://doi.org/10.1111/mafi.12203.

Eckstein, S. and Kupper, M. (2019)
Computation of optimal transport and related hedging problems via penalization and neural networks.
*Applied Mathematics and Optimization.*

Eckstein S., Kupper M. and Pohl M.
Robust risk aggregation with neural networks.
*arXiv preprint arXiv:1811.00304*, 2018.